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2010 J. Phys. A: Math. Theor. 43 075202

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# Hidden superconformal symmetry of the spinless Aharonov–Bohm system

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Received 4 October 2009

Published 1 February 2010

Online at [stacks.iop.org/JPhysA/43/075202](http://stacks.iop.org/JPhysA/43/075202)

## Abstract

A hidden supersymmetry is revealed in the spinless Aharonov–Bohm problem. The intrinsic supersymmetric structure is shown to be intimately related to the scale symmetry. As a result, a bosonized superconformal symmetry is identified in the system. Different self-adjoint extensions of the Aharonov–Bohm problem are studied in the light of this superconformal structure and interacting anyons. The scattering problem of the original Aharonov–Bohm model is discussed in the context of the revealed supersymmetry.

PACS numbers: 11.30.Pb, 11.25.Hf, 71.10.Pm

## 1. Introduction

The Aharonov–Bohm (AB) effect was discovered theoretically 50 years ago [1, 2]. Since that time, it found various experimental confirmations [3], and has been transformed into one of the most studied problems in planar physics [4–10]; for a good review we refer the reader to [11]. The AB effect underlies the dynamical realization of anyons [12–14], which are currently supposed to play the key role in the fractional Hall effect [15]. It appears in the analysis of cosmic strings [16, 17], and planar gravity [18, 19]. This effect also plays an important role in the physics of graphene and nanotubes [20–22].

In their original work [1], Aharonov and Bohm pointed out the importance of the vector potential in quantum theory. Unlike in classical mechanics, it has a direct impact on quantum dynamics even when the electromagnetic field vanishes everywhere in the regions accessible to a charged particle. Such a situation is realized when the magnetic flux penetrating

perpendicular to the plane is contained in finite regions bounded by an impenetrable barrier. As a limit case we can consider the model given by the vector potential

$$A^i = \frac{\alpha}{\hbar e} \epsilon^{ij} \frac{r^j}{r^2}, \quad \vec{r} = (x, y), \quad (1.1)$$

which corresponds to a singular flux that punctures the plane in the origin  $x = y = 0$ . In comparison with the free particle on the punctured plane, the physics is changed via a nontrivial phase that the wavefunction acquires when it is moving around the point where the flux dwells. This is the core of the AB effect.

In this work, we are going to testify this model in the presence of a *hidden supersymmetry* [23]. We will show that the Hamiltonian of a *spinless* charged particle moving in the presence of the vector potential (1.1),<sup>5</sup>

$$H_\alpha = \mathcal{P}_x^2 + \mathcal{P}_y^2 = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{1}{r^2} (-i\partial_\varphi + \alpha)^2, \quad (1.2)$$

$$\mathcal{P}_x = -i \partial_x - \alpha \frac{y}{r^2}, \quad \mathcal{P}_y = -i \partial_y + \alpha \frac{x}{r^2}, \quad (1.3)$$

$x = r \cos \varphi$ ,  $y = r \sin \varphi$ , possesses a rich algebraic structure of both exact (not dependent on time explicitly) and dynamical (time dependent) integrals of motion, that close for a superconformal superalgebra.

The key ingredients of a supersymmetric structure are supercharges  $Q_a$ , a Hamiltonian  $H$  and a grading operator  $\Gamma$ . The grading operator separates the set of relevant operators into families of bosonic and fermionic observables in accordance with whether they commute or anticommute with it. Supercharges are supposed to be fermionic while Hamiltonian is the bosonic operator,

$$\{\Gamma, Q_a\} = [H, \Gamma] = 0, \quad \Gamma^2 = 1. \quad (1.4)$$

We speak about a hidden supersymmetry when the operators  $Q_a$  and  $\Gamma$  can be found despite the lack of fermionic (spin) degrees of freedom in a system. The hidden supersymmetric structure has been observed in various physically interesting one-dimensional models, including the Dirac delta function potential problem, the reflectionless Pöschl–Teller system [24] and periodic finite-gap quantum systems [25, 26]. It was also observed in the bound-state Aharonov–Bohm effect [24] that corresponds to a particle confined to a circle. In those systems, the hidden supersymmetry reflects their peculiar spectral and scattering properties.

We will look for the operators  $\Gamma$ ,  $Q_1$  and  $Q_2$  that would satisfy (1.4) and

$$\{Q_a, Q_b\} = 2\delta_{ab} H_\alpha, \quad [H_\alpha, Q_a] = 0, \quad Q_a = Q_a^\dagger, \quad a, b = 1, 2. \quad (1.5)$$

These relations correspond to the Lie superalgebra of quantum mechanical  $N = 2$  supersymmetry<sup>6</sup>. The supercharges  $Q_1$  and  $Q_2$  can be nonlocal in general, as they correspond to the square roots of the spinless differential operator  $H_\alpha$ .

The Hamiltonian  $H_\alpha$  does not determine the dynamics of the particle uniquely until its actual domain of definition is fixed. The ambiguity in the proper definition of the system is intimately related to the self-adjoint extensions of the Hamiltonian. Physically, this corresponds to different possibilities of realizing the condition of impenetrability of the region  $x = y = 0$ . The task of self-adjoint extensions has been analyzed extensively in the literature. The case of a single magnetic vortex has been studied as a limit case of an

<sup>5</sup> We choose units in which the particle’s mass  $m = 1/2$  and  $\hbar = c = e = 1$ .

<sup>6</sup> In some systems, a hidden supersymmetry appears in a nonlinear form [27], in which the anticommutator of supercharges is a polynomial in the Hamiltonian [28].

impenetrable tube of finite radius with the internal magnetic field [29]. It was also analyzed directly with making use of the von Neumann theory of self-adjoint extensions [17, 30, 31].

Having in mind our objective, we cannot use these results directly as they do not contain any information on the existence of the supersymmetric structure described by (1.4) and (1.5). Our approach will be different: we will identify first the grading operator  $\Gamma$ , and construct operators  $Q_1$  and  $Q_2$  that will satisfy (1.4) and (1.5) formally. Then we will find their self-adjoint extensions. Hamiltonian, defined as the square of supercharges, will be self-adjoint by construction [32]. The obtained results will be compared with the known ones. As we will see, the self-adjoint extension with regular wavefunctions at the origin will be unitarily equivalent to the free particle system for integer values of the magnetic flux; meanwhile, it will match exactly with the model discussed by Aharonov and Bohm for non-integer values of  $\alpha$ . We also find two other self-adjoint extensions of  $H_\alpha$ , which for non-integer values of  $\alpha$  possess a hidden supersymmetry and correspond to supersymmetric two-anyon systems with contact interaction.

The work is organized as follows. In the next section, we construct a formal supercharge that satisfies the required properties. We then specify its self-adjoint extensions, and discuss the existence of  $N = 2$  supersymmetry in the system. Finding the eigenfunctions of the associated Hamiltonian, we show that the obtained system coincides with the original model discussed by Aharonov and Bohm. We analyze the action of supercharges on the wavefunctions to clarify whether we have a exact or spontaneously broken supersymmetry. In section 3, we consider two other self-adjoint extensions of  $H_\alpha$ , which possess a hidden supersymmetry. Particular attention is given to the case of the semi-integer flux in section 4, where an  $su(2)$  family of grading operators exists. In section 5, we discuss the conformal symmetry of the systems and confirm their scale invariance. Sequentially, we extend the algebraic structure of the hidden supersymmetry by conformal symmetry. In section 6, we provide an alternative interpretation of the model in terms of anyons. The last section is devoted to a brief summary and discussion of the results, with emphasis on their physical aspects. Particularly, we discuss the scattering problem in the original Aharonov–Bohm model in the light of the hidden supersymmetry and related translation symmetry breaking. We also list some open problems which would be interesting for a future research. Appendices include details on self-adjoint extensions of the supercharges considered in sections 2 and 3, and explicit formulas for the domains of the operators discussed in section 5.

## 2. Hidden $N = 2$ supersymmetry in the spinless AB system

In general, a formal Hamiltonian operator  $H_\alpha$  (1.2) admits a four-parametric  $U(2)$  family of self-adjoint extensions, which specify physically different configurations, distinct in their spectral and scattering properties [31, 33]. The spectrum depends strongly on the actual choice of the domain of definition of  $H_\alpha$ ; besides a continuous part of non-negative energy scattering states, it may contain up to two bound states of negative energy. As we stated above, our goal is to examine the model for the presence of the hidden supersymmetries (1.4), (1.5) generated by self-adjoint supercharges. This excludes immediately those self-adjoint extensions of (1.2) in which bound states are present, since negative energy levels would imply purely imaginary eigenvalues for the supercharges.

The general solution of the partial-wave stationary Schrödinger equation for non-negative energy  $E = k^2$ ,  $k \geq 0$ ,

$$H_\alpha \Psi_{k,l} = k^2 \Psi_{k,l}, \quad (2.1)$$

is a linear combination of Bessel,  $\mathcal{J}_{|l+\alpha|}(kr)$ , and Neumann,  $\mathcal{Y}_{|l+\alpha|}(kr)$ , functions multiplied by  $e^{il\varphi}$ . The concrete choice of the linear combination is specified uniquely by the domain of definition of the Hamiltonian. In their seminal work [1], Aharonov and Bohm considered the model where only regular at  $r = 0$  solutions were allowed, i.e. their solution of (2.1) was of the form

$$\Psi_{k,l} \sim \mathcal{J}_{|l+\alpha|}(kr) e^{il\varphi}. \tag{2.2}$$

This gives rise to a unique fixing of the self-adjoint extension of the operator  $H_\alpha$  that corresponds to the Aharonov–Bohm system, which we denote by  $H_\alpha^{AB}$ .

The aim of the present section is to reveal a hidden supersymmetry in the Aharonov–Bohm system. We proceed as follows: first of all, we identify the  $\mathbb{Z}_2$ -grading operator of the bosonized supersymmetry. Then we define a formal supercharge operator, find its self-adjoint extension and obtain the second odd generator of the  $N = 2$  supersymmetry. After that we show that the square of the found supercharges coincides with the Hamiltonian  $H_\alpha^{AB}$  of the Aharonov–Bohm system.

Consider a nonlocal operator of rotation in  $\pi$ ,

$$\mathcal{R} f(x, y) = f(-x, -y), \quad \text{or} \quad \mathcal{R} f(r, \varphi) = f(r, \varphi + \pi), \tag{2.3}$$

which is presented in terms of the total angular momentum  $J = -i\partial_\varphi + \alpha$  as

$$\mathcal{R} = e^{-i\alpha\pi} e^{i\pi J}. \tag{2.4}$$

It is a unitary, Hermitian involutive operator,  $\mathcal{R}^2 = 1$ , which commutes with Hamiltonian (1.2), and can be identified as the grading operator  $\Gamma$ . Consider a formal nonlocal differential operator

$$Q_\alpha = \mathcal{P}_x + i\mathcal{R}(\alpha)\mathcal{P}_y, \quad \text{where} \quad \mathcal{R}(\alpha) = \begin{cases} \mathcal{R}, & \alpha \in (-1, 0) \bmod 2, \\ \mathcal{R}, & \alpha \in \mathbb{Z}, \\ -\mathcal{R}, & \alpha \in (0, 1) \bmod 2. \end{cases} \tag{2.5}$$

This operator and operator  $i\mathcal{R}Q_\alpha$  satisfy formally relations

$$\{Q_\alpha, \mathcal{R}\} = \{i\mathcal{R}Q_\alpha, \mathcal{R}\} = 0, \quad \{Q_\alpha, i\mathcal{R}Q_\alpha\} = 0. \tag{2.6}$$

On the other hand, we have

$$\{Q_\alpha, Q_\alpha\} = \{i\mathcal{R}Q_\alpha, i\mathcal{R}Q_\alpha\} = 2H_\alpha + 2i\mathcal{R}[\mathcal{P}_x, \mathcal{P}_y]. \tag{2.7}$$

The commutator  $[\mathcal{P}_x, \mathcal{P}_y]$  is just the two-dimensional Dirac delta function. Unlike the one-dimensional case, such a term is not uniquely defined in the planar quantum systems [34, 35]. As was discussed in [36], the self-adjoint extension of the Hamiltonian  $H_\alpha$  has to be specified to define consistently the operator. When we specify the actual domain of the self-adjoint extension of  $H_\alpha$ , the Dirac delta function term is redundant in the potential since its manifestation can be understood in the asymptotic behavior of the wavefunctions near the origin<sup>7</sup>. In our current case, it will suffice to fix the self-adjoint extension of  $Q_\alpha$  since the square of a self-adjoint operator is self-adjoint as well.

Before we step to the analysis of the self-adjoint extension of  $Q_\alpha$ , let us note that the actual choice of the signs in definition of  $\mathcal{R}(\alpha)$  in (2.5) is crucial. An alternative choice of the sign for the same flux value case leads to a different self-adjoint extension of  $H_\alpha$ , and will be discussed in the next section. As we will see later in this section, the exception is in the case of integer flux values. For  $\alpha \in \mathbb{Z}$ , both choices  $\mathcal{R}(\alpha) = \mathcal{R}$  and  $\mathcal{R}(\alpha) = -\mathcal{R}$  lead to the same result.

<sup>7</sup> The same happens also in one dimension: when we require the wavefunction to be continuous at  $x = 0$  and specify its finite derivative jump there, the delta potential term can be omitted from the Hamiltonian operator [34].

The operator  $Q_\alpha$  (2.5) defined on the smooth functions with compact support is symmetric. Hence, the machinery of the von Neumann theory can be applied to find its self-adjoint extensions. It can be checked that  $Q_\alpha$  is essentially self-adjoint for any  $\alpha \in \mathbb{R}$ . Indeed, the equations  $(Q_\alpha)^\dagger f(r, \varphi) = \pm i f(r, \varphi)$  are not square integrable in  $\mathbb{R}^2$  solutions. The deficiency index is equal to  $(0, 0)$ , and the operator  $Q_\alpha$  does have a unique self-adjoint extension, its closure, which we denote as  $Q_\alpha^{AB}$ . Its domain of definition  $\mathcal{D}(Q_\alpha^{AB})$  is given by equations (A.10) and (A.11) in appendix A.

To play the role of the supercharge, the operator  $Q_\alpha^{AB}$  has to anticommute with the grading operator. The operator  $\mathcal{R}$  is essentially self-adjoint on  $\mathcal{D}(Q_\alpha^{AB})$  and leaves this space invariant. Hence, the anticommutation relation  $\{Q_\alpha^{AB}, \mathcal{R}\} = 0$  is well defined on  $\mathcal{D}(Q_\alpha^{AB})$ . This allows us to construct immediately the second self-adjoint supercharge  $i\mathcal{R}Q_\alpha^{AB}$ , defined on  $\mathcal{D}(Q_\alpha^{AB})$  as well. The square of the supercharges gives the self-adjoint Hamiltonian  $H_\alpha^c$  that is defined as

$$H_\alpha^c = (Q_\alpha^{AB})^2, \quad \mathcal{D}(H_\alpha^c) := \{\Phi \in \mathcal{D}(Q_\alpha^{AB}) \mid Q_\alpha^{AB}\Phi \in \mathcal{D}(Q_\alpha^{AB})\}. \quad (2.8)$$

Let us now show that the system described by  $H_\alpha^c$  coincides with the model proposed by Aharonov and Bohm. To do this, we will find eigenfunctions of  $H_\alpha^c$ .

To simplify the forthcoming analysis, let us comment on the relation between the systems  $H_\alpha^c$  and  $H_{\alpha+n}^c$  with magnetic flux values different in integer number  $n \in \mathbb{Z}$ . A simple formal operator equality

$$H_{\alpha+n} = U_n^{-1} H_\alpha U_n \quad (2.9)$$

suggests that the unitary transformation  $U_n = e^{in\varphi}$  is associated with the change in the magnetic flux of the system. It is indeed the case. First, we have  $U_n^{-1} \mathcal{R} U_n = (-1)^n \mathcal{R}$ , and for  $\alpha \notin \mathbb{Z}$  there holds

$$U_n^{-1} Q_\alpha^{AB} U_n = Q_{\alpha+n}^{AB}, \quad U_n^{-1} \mathcal{D}(Q_\alpha^{AB}) = \mathcal{D}(Q_{\alpha+n}^{AB}). \quad (2.10)$$

The case of  $\alpha \in \mathbb{Z}$  has, however, a peculiarity, and deserves a separate comment. For  $\alpha = n$ , there exists a system with hidden supersymmetry represented by self-adjoint operators  $H_n^c$  and  $Q_n^{AB}$  defined on corresponding domains. We can use the transformation  $U_1$  to construct another system with the same flux, described by  $H_n^c = U_1^{-1} H_{n-1}^c U_1$  and  $U_1^{-1} Q_{n-1}^{AB} U_1$ . For the transformed supercharge domain, there holds a relation

$$U_1^{-1} \mathcal{D}(Q_{n-1}^{AB}) = \mathcal{D}(Q_n^{AB}),$$

which means that the independent integrals of motion  $Q_n^{AB}$ ,  $\mathcal{R}$  and  $U_1^{-1} Q_{n-1}^{AB} U_1 = \mathcal{P}_x - i\mathcal{R}\mathcal{P}_y$  coexist in the same domain  $\mathcal{D}(Q_n^{AB})$ . Their linear combinations (including their multiplications by  $\mathcal{R}$ ) lead to another set of integrals of motion, given by  $\mathcal{R}$ ,  $\mathcal{P}_x$ ,  $\mathcal{P}_y$  and their multiples by  $\mathcal{R}$ . But  $\mathcal{P}_x$  and  $\mathcal{P}_y$  are the generators of translation in the plane, and, hence, the system described by  $H_n^c$  has a translational symmetry. As we will see,  $H_0^c$  corresponds to a free particle in the plane, which, of course, possesses translational invariance. Then the revealed translational symmetry of  $H_n^c$  can be understood as a consequence of the unitary equivalence of  $H_0^c$  and  $H_n^c$ .

Note that the Hamiltonian  $H_0^c$  is invariant, in addition, under spatial reflections. As there is no preferential direction in the plane, we can consider two reflections

$$\mathcal{R}_x g(x, y) \mathcal{R}_x = g(-x, y), \quad \mathcal{R}_y g(x, y) \mathcal{R}_y = g(x, -y), \quad (2.11)$$

which satisfy the relations

$$\mathcal{R}_x^2 = \mathcal{R}_y^2 = 1, \quad [\mathcal{R}_x, \mathcal{R}_y] = 0, \quad \mathcal{R} = \mathcal{R}_x \mathcal{R}_y. \quad (2.12)$$

In the polar coordinates their action is given by

$$\mathcal{R}_x f(r, \varphi) \mathcal{R}_x = f(r, \pi - \varphi), \quad \mathcal{R}_y f(r, \varphi) \mathcal{R}_y = f(r, -\varphi). \quad (2.13)$$

They commute with the operator  $\mathcal{R}$ , therefore they should be treated as nonlocal *even* integrals of motion within the supersymmetric structure. Despite their involutive nature, either of these two operators can be identified as the grading operator since they do not anticommute with the supercharge (2.5) (they do not commute with (2.5) either). As we will see in section 4, the twisted analogs of the operators (2.11) emerge nontrivially in the systems with a half-integer flux.

We conclude that the change in the sign of  $\mathcal{R}(\alpha)$  in definition (2.5) for an integer flux value case reduces to a unitary transformation, and that this sign ambiguity gives rise to the translational invariance of  $H_n^c$ . At the same time, we can see that the complete knowledge of the system for  $\alpha \in [-1, 0)$  (or for  $\alpha \in [0, 1)$ ) provides a detailed description for any other value of the magnetic flux as well. We will employ this fact in the forthcoming analysis of the spectral properties and supersymmetric structure of the system.

Let us fix the flux to be  $\alpha \in [-1, 0)$ . In the polar coordinates, the supercharge  $Q_\alpha^{AB}$  reads

$$Q_\alpha^{AB} = -\iota e^{i\varphi} \left[ \partial_r - \frac{1}{r}(-\iota \partial_\varphi + \alpha) \right] \Pi_- - \iota e^{-i\varphi} \left[ \partial_r + \frac{1}{r}(-\iota \partial_\varphi + \alpha) \right] \Pi_+, \quad (2.14)$$

where

$$\Pi_\pm = \frac{1}{2}(1 \pm \mathcal{R}) \quad (2.15)$$

are the projectors on the subspaces of even ( $\Pi_+$ ) and odd ( $\Pi_-$ ) partial waves. It preserves subspaces  $\mathcal{H}_l$ ,

$$\mathcal{H}_l := \mathcal{L}\{e^{i(2l-1)\varphi}, e^{i2l\varphi}\} \otimes \mathbf{L}_2(\mathbb{R}^+; r \, dr) \subset \mathbf{L}_2(\mathbb{R}^2), \quad l \in \mathbb{Z}, \quad (2.16)$$

where  $\mathcal{L}\{e^{i(2l-1)\varphi}, e^{i2l\varphi}\}$  is a linear space spanned by the indicated vectors. Then the eigenvalue problem can be solved separately in each  $\mathcal{H}_l$ .

The equation

$$Q_\alpha^{AB} \Phi_{l,\lambda} = \lambda \Phi_{l,\lambda} \quad \text{for} \quad \Phi_{l,\lambda} = \phi_{2l}(r) e^{i2l\varphi} + \phi_{2l-1}(r) e^{i(2l-1)\varphi} \quad (2.17)$$

is rewritten with the help of (2.14) in the form

$$\begin{aligned} \phi'_{2l}(r) + \frac{2l + \alpha}{r} \phi_{2l}(r) &= \iota \lambda \phi_{2l-1}(r), \\ \phi'_{2l-1}(r) + \frac{1 - (2l + \alpha)}{r} \phi_{2l-1}(r) &= \iota \lambda \phi_{2l}(r). \end{aligned} \quad (2.18)$$

The general solutions of (2.18) for nonzero eigenvalues  $\lambda$  are linear combinations of the Bessel functions of the first,  $\mathcal{J}_\nu(|\lambda|r)$ , and second,  $\mathcal{Y}_\nu(|\lambda|r)$ , kinds. The first is regular while the other one is singular at the origin, but both are not normalizable. To keep their interpretation in terms of scattering states, we require the wavefunctions not to have too strong divergence at infinity<sup>8</sup> and to respect the behavior near the origin, prescribed by the domain of definition. Since the singular solution violates the first requirement due to its divergence at  $r = 0$ , it has to be discarded. Then the acceptable solutions of (2.17) for  $\lambda \neq 0$  are

$$\Phi_{\lambda,l} \sim \mathcal{J}_{|2l+\alpha|}(|\lambda|r) e^{i2l\varphi} - i \frac{|\lambda|}{\lambda} \begin{cases} \mathcal{J}_{|1-2l-\alpha|}(|\lambda|r) e^{i(2l-1)\varphi} & \text{for } 2l + \alpha > 0, \\ -\mathcal{J}_{|1-2l-\alpha|}(|\lambda|r) e^{i(2l-1)\varphi} & \text{for } 2l + \alpha \leq 0. \end{cases} \quad (2.19)$$

The solutions of equations (2.18) for  $\lambda = 0$  with admissible behavior at the origin are

$$\Phi_{0,l} \sim \begin{cases} r^{2l+\alpha-1} e^{i(2l-1)\varphi} & \text{for } 2l + \alpha \geq 1, \\ r^{-2l-\alpha} e^{i2l\varphi} & \text{for } 2l + \alpha \leq 0. \end{cases} \quad (2.20)$$

<sup>8</sup> The mathematical framework for scattering states is provided by the rigged Hilbert space, where the functions can diverge at most as powers of  $r$  [37].

We pass now to the analysis of the eigenfunctions of  $H_\alpha^c$ . The Hamiltonian commutes with the generator of rotations since  $\mathcal{D}(H_\alpha^c)$  is invariant with respect to the action of  $J$ . Hence, one can find their common eigenfunctions  $\Psi_{|\lambda|,j}$ ,

$$H_\alpha^c \Psi_{|\lambda|,j} = \lambda^2 \Psi_{|\lambda|,j}, \quad J \Psi_{|\lambda|,l} = (l + \alpha) \Psi_{|\lambda|,l}. \quad (2.21)$$

They can be composed of the eigenvectors of  $Q_\alpha^{AB}$  corresponding to different signs of  $\lambda$ :

$$\begin{aligned} \Psi_{|\lambda|,2l} &\sim \Phi_{\lambda,l} + \Phi_{-\lambda,l} \sim \mathcal{J}_{|2l+\alpha|}(|\lambda|r) e^{2il\varphi}, \\ \Psi_{|\lambda|,2l-1} &\sim \Phi_{\lambda,l} - \Phi_{-\lambda,l} \sim \mathcal{J}_{|1-2l-\alpha|}(|\lambda|r) e^{i(2l-1)\varphi}. \end{aligned} \quad (2.22)$$

The zero-energy eigenstates of  $H_\alpha^c$  are

$$\Psi_{0,l} \sim r^{|l+\alpha|} e^{il\varphi}. \quad (2.23)$$

Note that the wavefunctions (2.22) vanish at the origin except in the special case of integer flux such that  $2l + \alpha = \beta \in \{0, 1\}$ . In this case,  $\mathcal{J}_{|\beta-2l-\alpha|}(|\lambda|r) = \mathcal{J}_0(|\lambda|r) \rightarrow 1$  for  $r \rightarrow 0$ , that is in agreement with the results on the self-adjoint extension of the free particle in the punctured plane [38]. The exclusion of the origin is of no importance here since the considered functions are regular at this point. In fact, the considered self-adjoint extension  $H_0^c$  of  $H_\alpha$  with  $\alpha = 0$  is in correspondence with the system of the free particle, since its domain of definition is spanned by the same complete basis of partial waves  $\mathcal{J}_{|m|}(kr) e^{im\varphi}$ .

We can compare the system represented by  $H_\alpha^c$  with the original setting of Aharonov and Bohm in a similar vein. The behavior of the wavefunctions near the origin is prescribed in the same way in both systems. This leads to the same complete basis of partial waves given by (2.19) and (2.20). Hence,  $H_\alpha^c$  and  $H_\alpha^{AB}$  represent the same self-adjoint extension of  $H_\alpha$ .

Thus, the system described by  $H_\alpha^c$  coincides with that discussed originally by Aharonov and Bohm:

$$H_\alpha^c = H_\alpha^{AB}. \quad (2.24)$$

This means that the Aharonov–Bohm model possesses the hidden  $N = 2$  supersymmetry generated by the supercharges  $Q_\alpha^{AB}$  and  $i\mathcal{R}Q_\alpha^{AB}$ , in which the role of the grading operator is played by the operator  $\mathcal{R}$ . This result is valid for any value of the magnetic flux.

Now, let us discuss the nature of the revealed supersymmetry, and the action of the supercharges. The spectrum of the operator  $H_\alpha^{AB}$  consists of the continuous part only, which covers non-negative real numbers. Any value of energy  $E$  is infinitely degenerate since there is an infinite set of linearly independent generalized wavefunctions (2.22) corresponding to the given energy  $E = \lambda^2$ . Let us discuss the action of the supercharges  $Q_\alpha^{AB}$  and  $i\mathcal{R}Q_\alpha^{AB}$ . The second supercharge interchanges the eigenfunctions of  $Q_\alpha^{AB}$  with different signs of  $\lambda \neq 0$ , i.e. there holds

$$i\mathcal{R}Q_\alpha^{AB} \Phi_{\lambda,l} \sim \Phi_{-\lambda,l}. \quad (2.25)$$

Consequently, with the direct use of this relation and (2.22), we can write

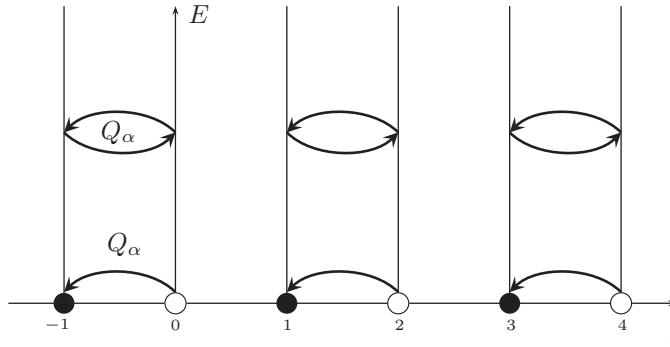
$$Q_\alpha \Psi_{|\lambda|,2l} \sim \Psi_{|\lambda|,2l-1}, \quad Q_\alpha \Psi_{|\lambda|,2l-1} \sim \Psi_{|\lambda|,2l}, \quad (2.26)$$

where  $Q_\alpha$  is  $Q_\alpha^{AB}$  or  $i\mathcal{R}Q_\alpha^{AB}$ .

The spectrum of  $H_\alpha^{AB}$  includes an infinitely degenerate zero-energy level. We restrict our consideration to the subspace  $\mathcal{H}_l$  where all the energy levels are doubly degenerate. This subspace is invariant under the action of the supercharges. Taking into account equation (2.20) for the zero modes of  $Q_\alpha^{AB}$ , we conclude that there exists just a single state in  $\mathcal{H}_l$  annihilated by  $Q_\alpha^{AB}$ . Fixing  $l \geq 0$ , we can write an explicit form of the involved functions:

$$H_\alpha^{AB} \Psi_{0,2l} = H_\alpha^{AB} \Psi_{0,2l-1} = 0, \quad Q_\alpha^{AB} \Psi_{0,2l} \sim \Psi_{0,2l-1}, \quad Q_\alpha^{AB} \Psi_{0,2l-1} = 0. \quad (2.27)$$





**Figure 1.** For  $\alpha \in [-1, 0) \bmod 2$ , the supercharges  $Q_\alpha \in \{Q_\alpha^{AB}, i\mathcal{R}Q_\alpha^{AB}\}$  preserve the subspaces  $\mathcal{H}_l$  defined in (2.16). We illustrate the action of the supercharges in these subspaces for  $l = 0, 1, 2$ . The zero-energy states (2.23) are represented by the circles: the black circles correspond to the zero modes of  $Q_\alpha$ . The arrows between the same energy levels in each  $\mathcal{H}_l^{AB}$  correspond to relations (2.26) for  $E > 0$  and to relations (2.27) for  $E = 0$ .

This resembles the Jordan block structure, which can appear in diagonalization of a finite-dimensional matrix. It does not contradict the self-adjointness of  $Q_\alpha^{AB}$ —the supercharge can be diagonalized by making use of its eigenstates (2.19) and (2.20).

Hence, the supercharges annihilate just half of the zero-energy states. The rest of these states is transformed into the kernel of the supercharges (see figure 1).

This picture can be compared with the cases of unbroken and broken supersymmetries in non-periodic one-dimensional systems. There, particularly, the unbroken supersymmetry is related to the existence of a singlet bound state of zero energy, annihilated by a supercharge. The second, nonphysical solution corresponding to zero energy is transformed to a physical one by a supercharge. In the present case, the continuous nature of the spectrum together with the infinite degeneracy of the energy levels prevents us from a similar classification of the revealed hidden supersymmetry. On the other hand, there is some similarity of the revealed hidden supersymmetric structure with that appearing in one-dimensional finite-gap periodic quantum systems, cf [26].

In conclusion of this section, let us make a few comments on the structure of the revealed supersymmetry, which later on will provide an alternative interpretation of the system in terms of anyons. In (2.15) we introduced projectors  $\Pi_\pm$  on the subspaces of even and odd orbital angular momenta. This allows us to separate the domain  $\mathcal{D}(H_\alpha^{AB})$  into two subsets  $\Pi_\pm \mathcal{D}(H_\alpha^{AB})$ , each of which consists of eigenvectors of  $\mathcal{R}$  with fixed eigenvalue +1 or -1. We can employ the matrix representation of the projectors,

$$\Pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.28}$$

The Hamiltonian  $H_\alpha^{AB}$  as well as other operators can be rewritten in the matrix form,

$$H_\alpha^{AB} = \begin{pmatrix} H_{\alpha,+}^{AB} & 0 \\ 0 & H_{\alpha,-}^{AB} \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.29}$$

where  $H_{\alpha,\pm}^{AB} = \Pi_\pm H_\alpha^{AB}$ . The supercharge  $Q_\alpha^{AB}$  is an antidiagonal operator and its explicit form for  $\alpha \in [-1, 0)$  can be deduced from (2.14) and (2.28). In this framework, the wavefunction  $\psi$  from the domain of  $H_\alpha^{AB}$  is just a column vector, whose upper element is composed of even partial waves,  $\psi_+ = \Pi_+ \psi = \sum_{l \in \mathbb{Z}} g_l^+(r) e^{i2l\varphi}$ , while the lower component consists of

odd partial waves,  $\psi_- = \Pi_- \psi = \sum_{l \in \mathbb{Z}} g_l^-(r) e^{i(2l-1)\varphi}$ . Such a representation reveals an obvious similarity of the hidden supersymmetry of the spinless Aharonov–Bohm system with the supersymmetry of a usual form, associated with the introduction of the spin degrees of freedom [39, 40].

### 3. Exotic models

The choice of the signs we made in the definition of  $\mathcal{R}(\alpha)$  in (2.5), and the observed ambiguity for the  $\alpha \in \mathbb{Z}$  case, led us to reveal the hidden  $N = 2$  supersymmetry in the original Aharonov–Bohm system. In this section we investigate the consequences of the alternative choice of the signs in (2.5) for non-integer flux values.

So, let us consider the operator

$$\tilde{Q}_\alpha = \mathcal{P}_x - i\mathcal{R}(\alpha)\mathcal{P}_y, \quad \text{where} \quad \mathcal{R}(\alpha) = \begin{cases} \mathcal{R} & \alpha \in (1, 2) \bmod 2, \\ -\mathcal{R} & \alpha \in (0, 1) \bmod 2. \end{cases} \quad (3.1)$$

The formal relations (2.6) and (2.7) imposed on the supercharge remain intact, up to the sign of the commutator term  $[\mathcal{P}_x, \mathcal{P}_y]$  in the square of  $\tilde{Q}_\alpha$ . This suggests that the difference, if any, could appear in self-adjoint extensions of the supercharge operator (3.1).

The transformation  $U_1$  changes the flux of the system in one unit. It maintains the self-adjointness of the operators, i.e. when an operator  $\mathcal{O}$  is self-adjoint on  $\mathcal{D}(\mathcal{O})$ , the operator  $\tilde{\mathcal{O}} = U_1^{-1}\mathcal{O}U_1$  is self-adjoint on  $U_1^{-1}\mathcal{D}(\mathcal{O})$ . This means that when we find all the admissible self-adjoint extensions of  $\tilde{Q}_\alpha$  for  $\alpha \in (-1, 0) \bmod 2$ , we can get all the self-adjoint extensions of the operator for  $\alpha \in (0, 1) \bmod 2$  just by the application of this transformation. The inverse is also true by changing the transformation  $U_1$  for  $U_1^{-1} = U_{-1}$ . Without loss of generality, we restrict our analysis to  $\alpha \in (0, 1) \bmod 2$ .

As the operator  $\tilde{Q}_\alpha$  for  $\alpha \in (0, 1) \bmod 2$  coincides formally with the operator  $Q_\alpha$  for  $\alpha \in (-1, 0) \bmod 2$ , we can use directly equation (2.14) to express the operator in polar coordinates, just keeping in mind the different range of  $\alpha$ . The operator  $\tilde{Q}_\alpha$  preserves the subspaces (2.16), and is symmetric on  $C_0^\infty(\mathbb{R}^2 - \{0\})$ . The domains of its conjugate and its closure are presented in appendix A.

We have to solve the deficiency equations  $\tilde{Q}_\alpha^\dagger \psi = \pm i\psi$  to reveal the bases of the deficiency subspaces. Relation (2.14) together with (2.16) simplifies this task since the problem can be inspected for each subspace  $\mathcal{H}_l$  separately. The deficiency indexes are vanishing again in all the subspaces  $\mathcal{H}_l$  except the subspace  $\mathcal{H}_{l_0}$  given by the integer  $l_0$  such that  $2l_0 + \alpha \in (0, 1)$ . In contrary to (2.5), the deficiency indexes of  $\tilde{Q}_\alpha$  are (1, 1), so that there exists a  $U(1)$  family of self-adjoint extensions  $\tilde{Q}_\alpha^\gamma$  of  $\tilde{Q}_\alpha$ . The detailed derivation of the result is rather technical (see appendix A), and we present the final form of the domain of the self-adjoint operator  $\tilde{Q}_\alpha^\gamma$ :

$$\mathcal{D}(\tilde{Q}_\alpha^\gamma) := \{\Phi(r, \varphi) = f(r, \varphi) + A[\Phi_+(r, \varphi) + e^{i\gamma}\Phi_-(r, \varphi)] \mid f(r, \varphi) \in \mathcal{D}(\overline{Q_\alpha}), A \in \mathbb{C}, \gamma \in [0, 2\pi)\}, \quad (3.2)$$

where  $\overline{Q_\alpha}$  is the closure of  $\tilde{Q}_\alpha$ . Expanding the function  $f(r, \varphi) \in \mathcal{D}(\overline{Q_\alpha})$  in partial waves  $f(r, \varphi) = \sum_l f_l(r) e^{il\varphi}$ , we find that the radial parts  $f_l(r)$  have to have the following asymptotic behavior near the origin:  $|f_l(r)| = O(1)$  for  $l \notin \{2l_0, 2l_0 - 1\}$ , while  $|f_{2l_0}(r)| = o(r^{-2l_0-\alpha})$  and  $|f_{2l_0-1}(r)| = o(r^{-1+2l_0+\alpha})$ . The functions  $\Phi_\pm(r, \varphi)$  form the basis of deficiency subspaces,  $\tilde{Q}_\alpha^\dagger \Phi_\pm(r, \varphi) = \pm i\Phi_\pm(r, \varphi)$ , and can be written in terms of McDonald functions

$$\Phi_\pm = K_{2l_0+\alpha}(r) e^{2l_0i\varphi} \pm K_{1-(2l_0+\alpha)}(r) e^{(2l_0-1)i\varphi}. \quad (3.3)$$

The operator  $\mathcal{R}$  is essentially self-adjoint on  $\mathcal{D}(\tilde{Q}_\alpha^\gamma)$ , but the requirement  $\{\tilde{Q}_\alpha^\gamma, \mathcal{R}\} = 0$  is consistent if and only if the operator  $\mathcal{R}$  leaves  $\mathcal{D}(\tilde{Q}_\alpha^\gamma)$  invariant. Using (3.2) and the fact that  $\mathcal{R}\mathcal{D}(\overline{Q}_\alpha) = \mathcal{D}(\overline{Q}_\alpha)$ , we get

$$\mathcal{R}(\mathcal{D}(\tilde{Q}_\alpha^\gamma)) = \mathcal{D}(\tilde{Q}_\alpha^{2\pi-\gamma}). \quad (3.4)$$

The requirement on the invariance of  $\mathcal{D}(\tilde{Q}_\alpha^\gamma)$  holds true for two values of parameter  $\gamma$  only:

$$\gamma = 0, \pi \pmod{2\pi}. \quad (3.5)$$

Hence, the  $N = 2$  supersymmetric structure is admissible just for these values of the parameter  $\gamma$ . If not stated otherwise, we will restrict  $\gamma \in \{0, \pi\}$  from now on. In this case, the domains of  $\tilde{Q}_\alpha^\gamma$  and  $i\mathcal{R}\tilde{Q}_\alpha^\gamma$  coincide. It is worth mentioning that both  $\mathcal{D}(\tilde{Q}_\alpha^0)$  and  $\mathcal{D}(\tilde{Q}_\alpha^\pi)$  acquire a particularly simple form,

$$\mathcal{D}(\tilde{Q}_\alpha^0) := \{\Phi(r, \varphi) = f(r, \varphi) + A K_{2l_0+\alpha}(r) e^{2il_0\varphi} | f(r, \varphi) \in \mathcal{D}(\overline{Q}_\alpha), A \in \mathbb{C}\}, \quad (3.6)$$

$$\mathcal{D}(\tilde{Q}_\alpha^\pi) := \{\Phi(r, \varphi) = f(r, \varphi) + A K_{1-2l_0-\alpha}(r) e^{(2l_0-1)i\varphi} | f(r, \varphi) \in \mathcal{D}(\overline{Q}_\alpha), A \in \mathbb{C}\}, \quad (3.7)$$

which manifests their invariance with respect to rotations generated by  $J$ .

The structure of  $N = 2$  supersymmetry is completed by the following definition of the self-adjoint Hamiltonian  $H_\alpha^\gamma$ :

$$H_\alpha^\gamma = (\tilde{Q}_\alpha^\gamma)^2, \quad \mathcal{D}(H_\alpha^\gamma) := \{\Phi \in \mathcal{D}(\tilde{Q}_\alpha^\gamma) | \tilde{Q}_\alpha^\gamma \Phi \in \mathcal{D}(\tilde{Q}_\alpha^\gamma)\}. \quad (3.8)$$

Hence, taking the different definition (3.1) of the supercharge, we reveal two distinct self-adjoint extensions  $H_\alpha^\gamma$  of the formal Hamiltonian operator  $H_\alpha$ , which, like the Aharonov–Bohm system considered in the previous section, are characterized by the hidden  $N = 2$  supersymmetry. As the domains of Hamiltonians are invariant with respect to  $J$ , the systems have rotational symmetry as well.

In the next step we will analyze the spectrum of  $H_\alpha^\gamma$  and find the associated wavefunctions. Since  $H_\alpha^\gamma$  is the square of the self-adjoint operator  $\tilde{Q}_\alpha^\gamma$ , and  $\mathcal{D}(H_\alpha^\gamma)$  is a subset of  $\mathcal{D}(\tilde{Q}_\alpha^\gamma)$ , we conclude that in correspondence with the hidden supersymmetric structure, the spectrum is non-negative.

We can employ equations (2.14), (2.17) and (2.18), keeping in mind the different range of  $\alpha$ ,  $2l_0 + \alpha \in (0, 1)$ ,  $l_0 \in \mathbb{Z}$ . Singular solutions of (2.18) have to be discarded in the subspaces  $\mathcal{H}_l$  for  $l \neq l_0$ . Hence, the eigenfunctions  $\Phi_{\lambda,l} = \phi_{2l} e^{2il\varphi} + \phi_{2l-1} e^{i(2l-1)\varphi}$  lying in these subspaces have exactly the same form as (2.19) and (2.20). The situation is different in the subspace  $\mathcal{H}_{l_0}$ . Due to (3.2), the admissible solutions  $\Phi_{\lambda,l_0}$  in  $\mathcal{H}_{l_0}$  have to fit the following asymptotic behavior near the origin:

$$\begin{aligned} \phi_{2l_0}(r) &= A(1 + e^{\gamma}) \frac{\Gamma(2l_0 + \alpha)}{2^{1-(2l_0+\alpha)}} r^{-2l_0-\alpha} + o(r^{-2l_0-\alpha}), \\ \phi_{2l_0-1}(r) &= A(1 - e^{\gamma}) \frac{\Gamma(1 - (2l_0 + \alpha))}{2^{2l_0+\alpha}} r^{-1+2l_0+\alpha} + o(r^{-1+2l_0+\alpha}), \end{aligned} \quad (3.9)$$

dictated explicitly by the relevant part  $A(\Phi_+(r, \varphi) + e^{i\gamma} \Phi_-(r, \varphi))$  of the domain of  $\tilde{Q}_\alpha^\gamma$ , where  $A$  is a constant. The solutions of (2.18) for  $\lambda \neq 0$  are

$$\begin{aligned} \phi_{2l_0}(r) &= C_1 \mathcal{I}_{|2l_0+\alpha|}(|\lambda|r) + C_2 \mathcal{Y}_{|2l_0+\alpha|}(|\lambda|r), \\ \phi_{2l_0-1}(r) &= -i \frac{|\lambda|}{\lambda} (C_1 \mathcal{I}_{2l_0+\alpha-1}(|\lambda|r) + C_2 \mathcal{Y}_{2l_0+\alpha-1}(|\lambda|r)), \end{aligned} \quad (3.10)$$

with the coefficients related to  $A$ ,

$$\begin{aligned} \frac{C_2}{A} &= -\frac{\pi}{2} \left(\frac{|\lambda|}{\mu}\right)^{2l_0+\alpha} (1 + e^{i\gamma}), \\ \frac{C_1}{A} &= \frac{\pi}{2 \sin(\pi(2l_0 + \alpha))} \\ &\times \left\{ \frac{i\lambda}{|\lambda|} \left(\frac{|\lambda|}{\mu}\right)^{1-2l_0-\alpha} (1 - e^{i\gamma}) + \cos(\pi(2l_0 + \alpha)) \left(\frac{|\lambda|}{\mu}\right)^{2l_0+\alpha} (1 + e^{i\gamma}) \right\}. \end{aligned} \tag{3.11}$$

The solution of (2.18) for  $\lambda = 0$  reads

$$\Phi_{l_0,0} \sim \begin{cases} r^{-2l_0-\alpha} e^{2il_0\varphi} & \text{for } \gamma = 0, \\ r^{-1+2l_0+\alpha} e^{(2l_0-1)i\varphi} & \text{for } \gamma = \pi. \end{cases} \tag{3.12}$$

Likewise in the previous section, there holds  $[H_\alpha^\gamma, J] = 0$  since  $\mathcal{D}(H_\alpha^\gamma)$  is invariant with respect to the action of  $J$ . Hence, one can find the common eigenfunctions  $\Psi_{|\lambda|,j}$ :

$$H_\alpha^\gamma \Psi_{|\lambda|,j} = \lambda^2 \Psi_{|\lambda|,j}, \quad J \Psi_{|\lambda|,j} = (j + \alpha) \Psi_{|\lambda|,j}. \tag{3.13}$$

They can be composed of the eigenvectors of  $\tilde{Q}_\alpha^\gamma$  which correspond to eigenvalues  $\pm\lambda$ . As long as  $l \neq l_0$ , the scattering states of  $H_\alpha^\gamma$  take the form (2.22). For  $l = l_0$ , we get

$$\Psi_{|\lambda|,2l_0} \sim \Phi_{\lambda,2l_0} + \Phi_{-\lambda,2l_0} \sim [\cos \pi \tilde{\alpha} \mathcal{J}_{\tilde{\alpha}}(|\lambda|r) - \sin \pi \tilde{\alpha} \mathcal{Y}_{\tilde{\alpha}}(|\lambda|r)] e^{2l_0i\varphi}, \quad \gamma = 0, \tag{3.14}$$

$$\Psi_{|\lambda|,2l_0-1} \sim \Phi_{\lambda,2l_0} - \Phi_{-\lambda,2l_0} \sim \mathcal{J}_{1-\tilde{\alpha}}(|\lambda|r) e^{i(2l_0-1)\varphi}, \quad \gamma = 0, \tag{3.15}$$

$$\Psi_{|\lambda|,2l_0} \sim \Phi_{\lambda,2l_0} - \Phi_{-\lambda,2l_0} \sim \mathcal{J}_{\tilde{\alpha}}(|\lambda|r) e^{2l_0i\varphi}, \quad \gamma = \pi, \tag{3.16}$$

$$\Psi_{|\lambda|,2l_0-1} \sim \Phi_{\lambda,2l_0} + \Phi_{-\lambda,2l_0} \sim [\cos \pi \tilde{\alpha} \mathcal{J}_{1-\tilde{\alpha}}(|\lambda|r) - \sin \pi \tilde{\alpha} \mathcal{Y}_{1-\tilde{\alpha}}(|\lambda|r)] e^{(2l_0-1)i\varphi}, \quad \gamma = \pi, \tag{3.17}$$

where  $\tilde{\alpha} = 2l_0 + \alpha$ . Like in (2.25), the second supercharge  $i\mathcal{R}\tilde{Q}_\alpha^\gamma$  interchanges eigenvectors of  $\tilde{Q}_\alpha^\gamma$  with different signs of  $\lambda$ . Consequently, the supercharges interchange the wavefunctions  $\Psi_{|\lambda|,2l}$  and  $\Psi_{|\lambda|,2l-1}$  given by (2.22) for  $l \neq l_0$ , and by (3.14)–(3.17) for  $l = l_0$ . In contrary to the Aharonov–Bohm Hamiltonian  $H_\alpha^{AB}$ , the operator  $H_\alpha^0$  (resp.  $H_\alpha^\pi$ ) has a singular zero mode  $\Psi_{0,2l_0} = r^{-2l_0-\alpha} e^{2l_0i\varphi}$  (resp.  $\Psi_{0,2l_0-1} = r^{-1+2l_0+\alpha} e^{(2l_0-1)i\varphi}$ ) in the subspace  $\mathcal{D}(H_\alpha^0) \cap \mathcal{H}_{l_0}$  (resp.  $\mathcal{D}(H_\alpha^\pi) \cap \mathcal{H}_{l_0}$ )<sup>9</sup>. However, there are no other differences in the analysis; the supercharge  $Q_\alpha^\gamma$  annihilates just half of the zero-energy states, mapping the rest to its kernel. Hence, the action of the supercharge  $Q_\alpha^\gamma$  is qualitatively in complete agreement with the discussion presented for  $Q_\alpha^{AB}$  in the previous section.

#### 4. Half-integer flux and twisted reflections

The operator  $\mathcal{R}$  commutes formally with  $H_\alpha$  for any value of the magnetic flux  $\alpha$ . In contrast, the reflection operators  $\mathcal{R}_x$  and  $\mathcal{R}_y$  defined in (2.11) are exclusive integrals of motion of the free particle. When the magnetic flux is switched on, they provoke a change in the sign of the magnetic flux in  $H_\alpha$ , i.e.  $\mathcal{R}_x H_\alpha \mathcal{R}_x = \mathcal{R}_y H_\alpha \mathcal{R}_y = H_{-\alpha}$ .

We can define the ‘twisted’ reflection operators  $\tilde{\mathcal{R}}_x = e^{i\alpha\pi} e^{-2i\alpha\varphi} \mathcal{R}_x$  and  $\tilde{\mathcal{R}}_y = e^{-2i\alpha\varphi} \mathcal{R}_y$ , for which formally  $[H_\alpha, \tilde{\mathcal{R}}_x] = [H_\alpha, \tilde{\mathcal{R}}_y] = 0$  and

$$\tilde{\mathcal{R}}_x^2 = \tilde{\mathcal{R}}_y^2 = 1. \tag{4.1}$$

<sup>9</sup> The zero modes of  $H_\alpha^\gamma$  as well as  $H_\alpha^{AB}$  can be understood as a low-energy limit of the properly normalized scattering states.

For a general value of  $\alpha$ , however, they do not preserve the space of  $2\pi$ -periodic functions.  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$  are defined consistently for  $\alpha = m$  or  $\alpha = m + \frac{1}{2}$  only, where  $m \in \mathbb{Z}$ . For  $\alpha = m$ , these operators are related to non-twisted reflections (2.11) by the unitary transformation  $U_m = e^{im\varphi}$ , and they commute, therefore, with  $\mathcal{R}$ . The situation is essentially different for half-integer values of  $\alpha$ . For  $\alpha = m + \frac{1}{2}$  we get

$$\tilde{\mathcal{R}}_x = -i e^{-i(2m+1)\varphi} \mathcal{R}_x, \quad \tilde{\mathcal{R}}_y = e^{-i(2m+1)\varphi} \mathcal{R}_y, \quad (4.2)$$

and

$$[\tilde{\mathcal{R}}_x, \tilde{\mathcal{R}}_y] = 2i\mathcal{R}, \quad [\mathcal{R}, \tilde{\mathcal{R}}_x] = 2i\tilde{\mathcal{R}}_y, \quad [\mathcal{R}, \tilde{\mathcal{R}}_y] = -2i\tilde{\mathcal{R}}_x, \quad (4.3)$$

where for the sake of convenience we included in the definition of  $\tilde{\mathcal{R}}_x$  an additional numerical factor  $(-1)^{m+1}$ . The operators satisfy also

$$\{\tilde{\mathcal{R}}_x, \tilde{\mathcal{R}}_y\} = \{\tilde{\mathcal{R}}_x, \mathcal{R}\} = \{\tilde{\mathcal{R}}_y, \mathcal{R}\} = 0. \quad (4.4)$$

Relations (4.3) and (4.4) mean that the twisted reflection operators (4.2) together with  $\mathcal{R}$  satisfy exactly the same set of algebraic relations as the three Pauli matrices, i.e. up to the numerical factor  $\frac{1}{2}$  they are generators of the spinorial representation of  $su(2)$ . In this section, we discuss the role of the triplet of reflection operators for the hidden supersymmetry of the systems with half-integer flux<sup>10</sup>.

Without loss of generality, set  $\alpha = 1/2$ . The operators  $\mathcal{R}$ ,  $\tilde{\mathcal{R}}_x = -ie^{-i\varphi} \mathcal{R}_x$  and  $\tilde{\mathcal{R}}_y = e^{-i\varphi} \mathcal{R}_y$  are symmetric on  $\mathcal{D}(Q_{1/2}^{AB})$ , or  $\mathcal{D}(\tilde{Q}_{1/2}^\gamma)$ ,  $\gamma \in [0, 2\pi)$ . In addition, neither  $\tilde{\mathcal{R}}_x f = \pm i f$  nor  $\tilde{\mathcal{R}}_y f = \pm i f$  have nontrivial solutions. The operators are essentially self-adjoint both on  $\mathcal{D}(Q_{1/2}^{AB})$  and on  $\mathcal{D}(\tilde{Q}_{1/2}^\gamma)$ . As  $\tilde{\mathcal{R}}_x^2 = \tilde{\mathcal{R}}_y^2 = 1$ , they are unitary as well. The described properties of the triplet of reflection operators allow us to introduce a three-parametric family of  $SU(2)$ -transformations

$$U(\beta_x, \beta_y, \beta) = e^{i(\beta_x \tilde{\mathcal{R}}_x + \beta_y \tilde{\mathcal{R}}_y + \beta \mathcal{R})}, \quad (4.5)$$

which will be important in the forthcoming analysis.

Consider now the Aharonov–Bohm model described by  $H_{1/2}^{AB}$ . The domain (A.10) of the supercharge  $Q_{1/2}^{AB}$  is invariant under the action of all the triplet of reflections:

$$\mathcal{R}\mathcal{D}(Q_{1/2}^{AB}) = \tilde{\mathcal{R}}_x \mathcal{D}(Q_{1/2}^{AB}) = \tilde{\mathcal{R}}_y \mathcal{D}(Q_{1/2}^{AB}) = \mathcal{D}(Q_{1/2}^{AB}). \quad (4.6)$$

Therefore, the set of integrals of motion of  $H_{1/2}^{AB}$  consisting of  $Q_{1/2}^{AB}$ ,  $i\mathcal{R}Q_{1/2}^{AB}$ ,  $\mathcal{R}$  and  $J$  has to be extended by the operators  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$ .

The question is then how they could be incorporated into the superalgebraic structure of the system. Keeping  $\mathcal{R}$  as the grading operator, the new integrals of motion are of the fermionic nature. Hence, the anticommutators with the supercharges  $Q_{1/2}^{AB}$  and  $i\mathcal{R}Q_{1/2}^{AB}$  should be computed, as well as their commutators with  $J$ . Both twisted reflections, however, anticommute with the angular momentum generator  $J$ . As a consequence, the repeated commutators with  $J$  give

$$[\tilde{\mathcal{R}}_x, J] = 2\tilde{\mathcal{R}}_x J, \quad [\tilde{\mathcal{R}}_x J, J] = 2\tilde{\mathcal{R}}_x J^2, \dots, \quad [\tilde{\mathcal{R}}_x J^n, J] = 2\tilde{\mathcal{R}}_x J^{n+1}, \quad (4.7)$$

and analogous relations for  $\tilde{\mathcal{R}}_y$ . Subsequent anticommutators of the odd integrals  $\tilde{\mathcal{R}}_x J^n$  and  $\tilde{\mathcal{R}}_y J^k$ ,  $n, k = 0, 1, \dots$ , produce the integrals of the form  $\mathcal{R}J^n$ . In the same way, the anticommutators of the twisted reflections with the supercharges  $Q_{1/2}^{AB}$  and  $i\mathcal{R}Q_{1/2}^{AB}$ , and corresponding repeated (anti)commutation relations reproduce the basic integrals multiplied by  $(H_{1/2}^{AB})^n$ . We see that the inclusion of the twisted reflection operators into the superalgebraic

<sup>10</sup> The special ‘magic’ of half-fluxons was discussed in a context different from the present one in [41, 42].

structure leads to its nonlinear deformation characterized by appearance of the multiplicative factors  $(J)^n$  and  $(H_{1/2}^{AB})^k$  in (anti)commutation relations, cf [24, 26–28].

Since the integrals  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$  satisfy relations (4.1), any of them can also be taken as the  $\mathbb{Z}_2$ -grading operator instead of  $\mathcal{R}$ . Such a possibility for an alternative choice of the grading operator resembles the tri-supersymmetric structure studied in [26]. The difference is that here the three involutive integrals mutually anticommute, while in the tri-supersymmetric structure analogous integrals mutually commute<sup>11</sup>. If, for instance,  $\tilde{\mathcal{R}}_x$  is identified as the grading operator, the operators  $Q_{1/2}^{AB}$  and  $i\tilde{\mathcal{R}}_x Q_{1/2}^{AB}$  will be nontrivial supercharges, and the angular momentum  $J$  has also to be treated as an odd generator. The anticommutator of the supercharge  $i\tilde{\mathcal{R}}_x Q_{1/2}^{AB}$  with  $\mathcal{R}$  generates then  $i\tilde{\mathcal{R}}_y Q_{1/2}^{AB}$  that has to be treated as an even integral. Further computing shows that with  $\mathcal{R}_x$  taken as the grading operator, we have, again, a nonlinear superalgebraic structure.

The picture is completely different in the case of the systems described by  $H_{1/2}^\gamma$  ( $\gamma \in \{0, \pi\}$ ). The operators  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$  are no longer symmetries of the system as the domain  $\mathcal{D}(H_\alpha^\gamma)$  is not invariant under their action. The unitary transformations (4.5) can be used to map the system  $H_{1/2}^\gamma$  ( $\gamma \in \{0, \pi\}$ ) to another, equivalent one, with the same supersymmetric structure. Let us discuss a few particular examples, where the transformed grading operator acquires particularly simple form. We are interested in the mappings which would interchange the operators  $\mathcal{R}$ ,  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$  in the role of the grading operator,

$$\mathcal{R} = -\mathcal{U}_0 \mathcal{R} \mathcal{U}_0^\dagger, \quad \tilde{\mathcal{R}}_x = -\mathcal{U}_1 \mathcal{R} \mathcal{U}_1^\dagger = \mathcal{U}_3 \mathcal{R} \mathcal{U}_3^\dagger, \quad \tilde{\mathcal{R}}_y = -\mathcal{U}_2 \mathcal{R} \mathcal{U}_2^\dagger = \mathcal{U}_4 \mathcal{R} \mathcal{U}_4^\dagger, \quad (4.8)$$

where the explicit form of the  $SU(2)$ -transformations is

$$\begin{aligned} \mathcal{U}_0 &= e^{i\frac{\pi}{2}\tilde{\mathcal{R}}_y} = i\tilde{\mathcal{R}}_y, \\ \mathcal{U}_1 &= e^{i(\frac{\pi}{2}\mathcal{R} - \frac{\pi}{4}\tilde{\mathcal{R}}_y)} = \frac{i}{\sqrt{2}}\mathcal{R}(1 - i\tilde{\mathcal{R}}_y), \quad \mathcal{U}_3 = e^{-i\frac{\pi}{4}\tilde{\mathcal{R}}_y} = \frac{1}{\sqrt{2}}(1 - i\tilde{\mathcal{R}}_y), \\ \mathcal{U}_2 &= e^{-i\frac{\pi}{4}\tilde{\mathcal{R}}_x} = \frac{1}{\sqrt{2}}(1 - i\tilde{\mathcal{R}}_x), \quad \mathcal{U}_4 = e^{i(\frac{\pi}{2}\mathcal{R} - \frac{\pi}{4}\tilde{\mathcal{R}}_x)} = \frac{i}{\sqrt{2}}\mathcal{R}(1 - i\tilde{\mathcal{R}}_x). \end{aligned} \quad (4.9)$$

Let us note that the transformations (4.8) together with (2.4) suggest that the twisted reflections  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$  can be written formally in the following way:

$$-\tilde{\mathcal{R}}_x = \exp(i\pi\mathcal{U}_3 J \mathcal{U}_3^\dagger), \quad \tilde{\mathcal{R}}_y = \exp(i\pi\mathcal{U}_1 J \mathcal{U}_1^\dagger). \quad (4.10)$$

Inspect now how these transformations change the other constituents of the supersymmetry, Hamiltonian and supercharges. Formally, the Hamiltonian  $H_{1/2}$  commutes with any of  $\mathcal{R}$ ,  $\tilde{\mathcal{R}}_x$  or  $\tilde{\mathcal{R}}_y$  so that it is invariant with respect to the  $SU(2)$ -transformations (4.5). The formal operator  $\tilde{Q}_{1/2}$  is transformed as

$$\begin{aligned} \mathcal{U}_0 \tilde{Q}_{1/2} \mathcal{U}_0^\dagger &= \tilde{Q}_{1/2}, \quad \mathcal{U}_1 \tilde{Q}_{1/2} \mathcal{U}_1^\dagger = -\tilde{Q}_{1/2}, \quad \mathcal{U}_3 \tilde{Q}_{1/2} \mathcal{U}_3^\dagger = \tilde{Q}_{1/2}, \\ \mathcal{U}_2 \tilde{Q}_{1/2} \mathcal{U}_2^\dagger &= i\tilde{\mathcal{R}}_x \tilde{Q}_{1/2}, \quad \mathcal{U}_4 \tilde{Q}_{1/2} \mathcal{U}_4^\dagger = i\tilde{\mathcal{R}}_y \tilde{Q}_{1/2}. \end{aligned} \quad (4.11)$$

Let us suppose that we take the self-adjoint extension  $H_{1/2}^0$  (with the supercharge  $\tilde{Q}_{1/2}^0$ ) as the initial system. The transformations (4.5) are unitary, and applied to  $\tilde{Q}_\alpha^0$  produce self-adjoint operators, defined on  $\mathcal{U}_k \mathcal{D}(\tilde{Q}_\alpha^0)$  for  $k \in \{0, 1, 2, 3, 4\}$ . The transformed supercharges in the upper line of (4.11) coincide formally with  $\tilde{Q}_{1/2}$ . As we found in the previous section, there exists a one-parametric family of the self-adjoint extensions of this operator, labeled  $\gamma$ . Hence,

<sup>11</sup> A supersymmetric structure with three mutually anticommuting involutive integrals of motion was observed recently in Bogolyubov–de Gennes system [43].

the systems produced by  $\mathcal{U}_0, \mathcal{U}_1$  and  $\mathcal{U}_3$  should fit into this classification scheme. This is indeed the case: we can write

$$\mathcal{U}_0 \tilde{\mathcal{Q}}_{1/2}^0 \mathcal{U}_0^\dagger = \tilde{\mathcal{Q}}_{1/2}^\pi, \quad \mathcal{U}_1 \tilde{\mathcal{Q}}_{1/2}^0 \mathcal{U}_1^\dagger = \tilde{\mathcal{Q}}_{1/2}^{\pi/2}, \quad \mathcal{U}_3 \tilde{\mathcal{Q}}_{1/2}^0 \mathcal{U}_3^\dagger = \tilde{\mathcal{Q}}_{1/2}^{3\pi/2}, \quad (4.12)$$

where the value of the index  $\gamma$  coherently reflects the domain of definition, given by (3.2). The remaining systems with the supercharges of the lower line in (4.11) do not belong to the family of self-adjoint operators  $\tilde{\mathcal{Q}}_{1/2}^\gamma$  as neither of the supercharges coincides formally with  $\tilde{\mathcal{Q}}_{1/2}$ . The explicit form of the domains of definition of the new supercharges for  $k \in \{1, 2, 3, 4\}$  can be written in the following compact form:

$$\mathcal{U}_k \mathcal{D}(\tilde{\mathcal{Q}}_{1/2}^0) = \{\Phi(r, \varphi) = f(r, \varphi) + A K_\alpha(r)(1 + i^k e^{-i\varphi}) | f(r, \varphi) \in \mathcal{D}(\overline{\mathcal{Q}}_\alpha), A \in \mathbb{C}\}. \quad (4.13)$$

Hence, for the semi-integer values of the magnetic flux  $\alpha$  we have a three-parametric family of the systems with hidden supersymmetry, associated with the formal supercharge operator

$$\mathcal{Q}_\alpha(\beta_x, \beta_y, \beta) = \mathcal{U} \tilde{\mathcal{Q}}_\alpha^0 \mathcal{U}^\dagger, \quad \mathcal{D}(\mathcal{Q}_\alpha(\beta_x, \beta_y, \beta)) = \mathcal{U} \mathcal{D}(\tilde{\mathcal{Q}}_\alpha^0), \quad \mathcal{U} = \mathcal{U}(\beta_x, \beta_y, \beta). \quad (4.14)$$

These systems fit into the general scheme of the self-adjoint extensions of the Aharonov–Bohm model discussed in [31], where the self-adjoint extensions of  $H_\alpha$  with broken rotational symmetry were observed. Despite the rotational symmetry being broken in our present case as well (see (4.13)), domains of definition are invariant with respect to the operator  $\tilde{\mathcal{J}}(\beta_x, \beta_y, \beta) = \mathcal{U} \mathcal{J} \mathcal{U}^\dagger$ , i.e. the systems associated with (4.14) are unitarily equivalent to the systems with rotational symmetry.

### 5. Superconformal symmetry

Jackiw showed that like a charge–monopole system [44], the original Aharonov–Bohm model is characterized by a dynamical conformal  $so(2, 1)$  symmetry [45]. We revealed the hidden  $N = 2$  supersymmetry not only in the Aharonov–Bohm system characterized by a regular behavior of the wavefunctions at the origin but also in exotic models corresponding to some special cases of the  $U(2)$  family of self-adjoint extensions of the formal Hamiltonian operator (1.2). On the other hand, if we look at the  $U(2)$  family of the self-adjoint extensions requiring the scale symmetry, this also excludes immediately those cases which are characterized by the presence of the bound states. Such a similarity with restrictions imposed by the requirement of the presence of the hidden supersymmetry, certainly, is worth a more in-depth look. In this section, we study the question of compatibility of the revealed hidden supersymmetric structure with the dynamical conformal symmetry.

Besides the Hamiltonian of the system, which we denote here by  $H$ , the dynamical conformal symmetry [46] is generated by the operators  $D$  and  $K$  that depend explicitly on time. They satisfy equation  $\frac{d}{dt} \mathcal{C} = \partial_t \mathcal{C} - i[H, \mathcal{C}] = 0$ ,  $\mathcal{C} = D, K$ , and their explicit form is given by

$$D = tH - \frac{1}{4}(\vec{x}\vec{P} + \vec{P}\vec{x}), \quad K = -2t^2H + 4tD + \frac{1}{2}\vec{x}^2. \quad (5.1)$$

The operator  $D$  generates dilatations, while  $K$  is the generator of the special conformal transformations. The conformal algebra  $so(2, 1)$  is established by the formal commutation relations

$$[D, K] = iK, \quad [H, K] = 4iD, \quad [H, D] = iH. \quad (5.2)$$

The domain  $\mathcal{D}_c$  where the commutators are well defined has to be specified. It has to be located at the intersection of the domains of all the involved operators. Also, the action of each of

the operators  $H$ ,  $K$  and  $D$  has to keep the wavefunction in the domains of the two remaining operators. For  $t = 0$ , the explicit form of  $D$  and  $K$  in polar coordinates is

$$D = \frac{i}{2}(1 + r\partial_r), \quad K = \frac{1}{2}r^2. \quad (5.3)$$

Both these operators are essentially self-adjoint. Indeed, the solutions of  $Df(r, \varphi) = \pm if(r, \varphi)$  are not square integrable, while deficiency equations  $Kf(r, \varphi) = \pm if(r, \varphi)$  do not have solutions at all. The domains of the essentially self-adjoint operator  $K$ , and of the self-adjoint operator  $\overline{D}$ , are described in appendix B. Fixing  $H$  to be one of the operators  $H_\alpha^{AB}$  or  $H_\alpha^\gamma$ ,  $\gamma = 0, \pi$ , we can write

$$\mathcal{D}_c = \{\Phi(r, \varphi) \in \mathcal{D}(H) \cap \mathcal{D}(\overline{D}) \cap \mathcal{D}(K) | H\Phi \in \mathcal{D}(\overline{D}) \cap \mathcal{D}(K), \overline{D}\Phi \in \mathcal{D}(H) \cap \mathcal{D}(K), K\Phi \in \mathcal{D}(\overline{D})\}. \quad (5.4)$$

This set is dense in  $L_2(\mathbb{R}^2)$  as it contains smooth functions with compact support ( $C_0^\infty(\mathbb{R}^2)$ ).

The generalized eigenvectors (scattering states) of  $H$  do not have compact support, and are not square integrable. However, they can serve to construct the wave packets which are normalizable, and represent physical states. These square integrable functions inherit some of the properties of the scattering states; they do not belong to  $C_0^\infty(\mathbb{R}^2)$ , and have a specific behavior of partial waves near the origin, dictated by  $\mathcal{D}(H)$ . We can ask whether they are present in  $\mathcal{D}_c$ . The necessary condition is that the operators  $K$  and  $D$  do not alter the asymptotic behavior of the partial waves near the origin.

The domain of definition of either  $H_\alpha^{AB}$  or  $H_\alpha^\gamma$  is rotationally invariant. The partial waves near the origin may not be more divergent than a fixed power of  $r$ , prescribed by the domain of definition. Keeping in mind the explicit form (5.3), we see that neither  $K$  nor  $D$  violate this restriction on the asymptotic behavior of partial waves. Hence, the domain  $\mathcal{D}_c$  includes the physically interesting states<sup>12</sup> composed of the scattering states. This conclusion is not evident for other self-adjoint extensions  $H_\alpha^\gamma$  when a general value of  $\gamma$  is considered. Let us just note that the invariance with respect to  $D$  is broken in general. The scale invariance is recovered for  $\gamma = 0$  or  $\gamma = \pi$  when  $\alpha$  is treated as a free parameter. For a fixed value of the magnetic flux  $\alpha = 1/2 \bmod 1$ , the scale symmetry appears in the whole family of self-adjoint extensions  $H_{1/2}^\gamma$  for any value of  $\gamma$ , see appendix B. Therefore, the hidden supersymmetry of the systems represented by  $H_\alpha^{AB}$  and  $H_\alpha^\gamma$  comes hand in hand with conformal symmetry and the scale invariance in particular. Below we show that both structures are compatible in the Lie algebraic sense, and give rise to the superconformal  $osp(2|2)$  symmetry.

The operators  $K$  and  $\overline{D}$  commute with  $\mathcal{R}$ , their domains are invariant with respect to the action of  $\mathcal{R}$  and they can be treated as bosonic generators in the framework of the extended superalgebra. The relevant commutation and anticommutation relations have to be computed to verify that the superalgebra is closed. The computation does not depend on the actual choice of the self-adjoint extension, so that we adopt the notation  $H$  for  $H_\alpha^{AB}$  or  $H_\alpha^\gamma$ , and, respectively,  $Q_1 = \mathcal{P}_x + i\varepsilon\mathcal{R}\mathcal{P}_y$  for  $Q_\alpha^{AB}$  or  $\tilde{Q}_\alpha^\gamma$ , and  $Q_2 = -i\varepsilon\mathcal{R}Q_1$ , where  $\varepsilon = +1$  or  $-1$  in dependence on the value of the flux  $\alpha$ , see equations (2.5) and (3.1). The self-adjoint generator of dilatations is denoted below by  $D$ . To close the superalgebra, two additional integrals of motion (explicitly dependent on time) have to be involved. In the commutator of  $K$  and  $Q_j$ , there appear new integrals of motion

$$[Q_j, K] = -iS_j, \quad S_2 = -i\varepsilon\mathcal{R}S_1, \quad (5.5)$$

where

$$S_1 = X + i\varepsilon\mathcal{R}Y, \quad X = x - 2t\mathcal{P}_x, \quad Y = y - 2t\mathcal{P}_y.$$

<sup>12</sup> We have in mind a two-dimensional exponentially decreasing (gaussian) wave packet for instance.



The mixed anticommutator of  $Q_j$  and  $S_k$  brings a new conserved quantity,  $\{Q_1, S_2\} = 2F$ ,

$$F = \varepsilon \mathcal{R} - J. \quad (5.6)$$

Completing the remaining relations dictated by the superalgebra, we end up with

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}H, & \{S_i, S_j\} &= 4\delta_{ij}K, \\ \{Q_j, S_k\} &= -4\delta_{jk}D + 2\varepsilon_{jk}F, \\ [Q_j, K] &= -iS_j, & [S_j, K] &= 0, \\ [Q_j, D] &= \frac{i}{2}Q_j, & [S_j, D] &= -\frac{i}{2}S_j, \\ [Q_j, H] &= 0, & [S_j, H] &= 2iQ_j, \\ [F, Q_j] &= i\varepsilon_{jk}Q_k, & [F, S_j] &= i\varepsilon_{jk}S_k, \\ [F, H] &= [F, K] = [F, D] = 0, \\ [F, \mathcal{R}] &= [H, \mathcal{R}] = [D, \mathcal{R}] = [K, \mathcal{R}] = \{Q_j, \mathcal{R}\} = \{S_j, \mathcal{R}\} = 0. \end{aligned} \quad (5.7)$$

Instead of the even generators  $J$  and  $\mathcal{R}$ , in addition to the linear combination (5.6) we define the operator

$$\mathcal{Z} = J - \frac{\varepsilon}{2}\mathcal{R}, \quad (5.8)$$

which commutes with all the other even and odd generators of superalgebra, playing the role of its central charge. The introduced operators  $S_j$ ,  $F$  and  $\mathcal{Z}$  are essentially self-adjoint on their natural domains of definition, see appendix B. Note that from the relation  $J = F + 2\mathcal{Z}$  it follows that  $Q_i$  and  $S_i$  are vector operators.

Likewise in the case of the conformal symmetry, the actual domain of definition  $\mathcal{D}_{sc}$  has to be specified to make the relations (5.2) and (5.7) consistent. It has to be an intersection of the domains of the involved operators (just let us recall that  $\mathcal{D}(Q_1) = \mathcal{D}(Q_2)$  and  $\mathcal{D}(S_1) = \mathcal{D}(S_2)$ ), and the action of any of them has to keep the function in the intersection of the domains of the remaining operators.

The same analysis applies as in the case of  $\mathcal{D}_c$ . The domain  $\mathcal{D}_{sc}$  is dense in  $L_2(\mathbb{R}^2)$  as it contains the set of smooth functions with compact support. We require that neither of the operators violates asymptotics of the functions near the origin—they should maintain or increase the power of the leading term in the asymptotic expansion. This requirement is met by all the new operators  $S_j$ ,  $F$  and  $\mathcal{Z}$ . Hence, the domain  $\mathcal{D}_{sc}$  can support physically interesting states represented particularly by wave packets.

We conclude that the three self-adjoint extensions  $H_\alpha^{AB}$ ,  $H_\alpha^0$  and  $H_\alpha^\pi$  possess the scale invariance as a consequence of their conformal symmetry. The conformal and hidden supersymmetric structures of these systems are compatible, and lead to the superconformal symmetry. The resulting algebraic structure corresponds to the superalgebra  $osp(1|2) \times o(2)$ , which was observed earlier in various physical models [47–50], including a spin-1/2 particle in the presence of a magnetic vortex [50]<sup>13</sup>. For spin-1/2 particle systems possessing the superconformal symmetry, the role of the grading operator is played by the matrix  $\sigma_3$ . We revealed here the same superalgebraic structure in the system without fermionic degrees of freedom.

<sup>13</sup> The analysis of the algebraic structure was performed in [50] on a formal level, without touching the questions of self-adjointness of corresponding generators.

## 6. Hidden supersymmetry and anyons

In early 1980s, Wilczek proposed a dynamical mechanism for the realization of anyons that is based on the Aharonov–Bohm effect [13]. Here we show that the anyon picture provides a rather natural interpretation for the hidden supersymmetric structure described in the previous sections.

Consider a two-anyon, planar system described by the formal Hamiltonian operator

$$H_{\text{any}} = 2 \sum_{l=1}^2 (\vec{p}_l - \vec{a}_l(\vec{r}))^2, \quad (6.1)$$

where  $\vec{p}_l = -i\partial/\partial\vec{x}_l$ ,  $\vec{r}$  is a relative coordinate,  $\vec{r} = \vec{x}_1 - \vec{x}_2$ , and we set the masses of the constituents  $m_1 = m_2 = 4$ . The constituent point particles are ‘statistically charged’, and each carries a ‘magnetic’ vortex described by the statistical vector potential:

$$a_l^k(\vec{r}) = -a_2^k(\vec{r}) = \frac{1}{2}\alpha\epsilon^{kl}\frac{r^l}{r^2}. \quad (6.2)$$

In the center of mass reference frame, Hamiltonian (6.1) takes the form (1.2).

The two-anyon system may be composed of statistically interacting identical bosons, or fermions. This means that when the statistical interaction is switched off ( $\alpha = 0$ ), the constituent particles should obey either Bose, or Fermi statistics. The identity of the particles and their statistics are taken into account by requiring that under the exchange,  $\vec{r} \rightarrow -\vec{r} \Rightarrow \varphi \rightarrow \varphi + \pi$ , the wavefunction obeys the relation  $\psi_{\alpha=0}(r, \varphi + \pi) = \eta\psi_{\alpha=0}(r, \varphi)$  with  $\eta = +1$  for boson, or  $\eta = -1$  for fermion constituents. This requirement is maintained when the statistical interaction is switched on. Therefore, we have

$$\psi_{\alpha}(r, \varphi) = \sum_l e^{il\varphi} f_{\alpha,l}(r), \quad l \in \begin{cases} 2\mathbb{Z} & \text{for anyons based on bosons,} \\ 2\mathbb{Z} + 1 & \text{for anyons based on fermions.} \end{cases} \quad (6.3)$$

Requiring Hamiltonian (6.1) to be a self-adjoint operator, its domain of definition has to be specified. The nontrivial behavior of the wavefunctions (6.3) near the origin, dictated by the particular choice of the self-adjoint extension, is then interpreted as a contact (zero-range) interaction of the anyons [51].

The anyon framework provides an interesting interpretation for the systems we studied in sections 2 and 3. As follows from the discussion at the end of section 2, Hamiltonians  $H_{\alpha}^{AB}$ ,  $H_{\alpha}^0$  and  $H_{\alpha}^{\pi}$  can be described by diagonal two-by-two matrices in representation where the grading operator  $\mathcal{R}$  is given by the Pauli matrix  $\sigma_3$ , see (2.29). The upper and lower components of the states in this representation correspond to  $\pi$ -periodic and  $\pi$ -antiperiodic in  $\varphi$  parts of the wavefunctions. Due to the described correspondence between Hamiltonian (6.1) of the two-anyon system and Hamiltonian (1.2), we conclude that the diagonal components  $\Pi_{\pm}H_{\alpha}^{AB}$  and  $\Pi_{\pm}H_{\alpha}^{\gamma}$  of the studied spinless systems can be understood as the Hamiltonians that describe the relative motion of the two-anyon systems. The upper component represents the system based on bosons (as it acts on  $\pi$ -periodic functions) while the lower diagonal operators rule the dynamics of the system based on fermions. The self-adjoint Hamiltonians  $\Pi_{\pm}H_{\alpha}^{\gamma}$  imply additional contact interaction of the anyons.

Therefore, the *hidden* superconformal symmetry that we revealed in the spinless Aharonov–Bohm system corresponds to an *explicit* center-of-mass supersymmetric structure of the system composed of the *two* two-anyon systems based on bosons *and* on fermions.

## 7. Discussion and outlook

Let us summarize and discuss the main results by stressing the physical aspects that are behind the revealed hidden supersymmetric structure.

For integer values of the flux,  $\alpha = n$ ,  $n \in \mathbb{Z}$ , the Aharonov–Bohm system is unitary equivalent to a planar free particle system ( $\alpha = 0$ ). The latter possesses the rotational and translational symmetries generated by the angular momentum operator  $J$ , and by mutually commuting momenta operators  $\mathcal{P}_x$  and  $\mathcal{P}_y$ . In correspondence with this, the Hamiltonian operator (1.2) can be factorized as

$$H_n = (\mathcal{P}_x + i\epsilon\mathcal{P}_y)(\mathcal{P}_x - i\epsilon\mathcal{P}_y), \quad (7.1)$$

or, alternatively, can be presented as a perfect square,

$$H_n = (\mathcal{P}_x + i\epsilon\mathcal{R}\mathcal{P}_y)^2, \quad (7.2)$$

where the parameter  $\epsilon$  can take any of two values,  $+1$  or  $-1$ , and  $\mathcal{R}$  is a nonlocal operator of rotation for angle  $\pi$ . For  $\alpha \neq n$ , the formal Aharonov–Bohm Hamiltonian (1.2) can also be factorized in the form (7.1), or (7.2). However, in the case of a non-integer flux, the operators  $\mathcal{P}_x$  and  $\mathcal{P}_y$  are not physical, and the translation invariance is broken, see below. Thus, for  $\alpha \neq n$ , (7.1) is a purely *formal* factorization. In contrast with (7.1), representation (7.2) can be well defined. A nontrivial property associated with factorization (7.2) is that for a given flux  $\alpha \neq n$ ; two different choices for the value of the parameter  $\epsilon$  correspond to physically distinct systems. For  $\alpha \in (-1, 0) \bmod 2$ ,  $\epsilon = +1$ , and  $\alpha \in (0, 1) \bmod 2$ ,  $\epsilon = -1$ , factorization (7.2) corresponds to the original system  $H_\alpha^{AB}$  investigated by Aharonov and Bohm [1, 2], which is characterized by a regular at the origin behavior of the Hamiltonian eigenfunctions. An alternative choice of the values of the parameter  $\epsilon$  in (7.2) gives rise to two different, *exotic* models given by self-adjoint Hamiltonians  $H_\alpha^\gamma$  with  $\gamma = 0, \pi$ , which are characterized by a singular behavior at the origin of their eigenfunctions in one specific partial wave correlated with the value of the flux, see equations (3.14)–(3.17). For half-integer values  $\alpha = n + 1/2$ , both exotic systems with  $\gamma = 0$  and  $\gamma = \pi$  are unitary equivalent, and like the Aharonov–Bohm model  $H_\alpha^{AB}$ , they possess additional nonlocal integrals of motion in the form of the twisted reflection operators  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$ . These nonlocal integrals together with  $\mathcal{R}$  satisfy the same algebraic relations as the three Pauli matrices, i.e., generate a spinorial representation of  $su(2)$  realized on the states of the corresponding system.

Identifying the nonlocal operator  $\mathcal{R}$  as the  $\mathbb{Z}_2$ -grading operator, we interpret the self-adjoint operator appearing in factorization (7.2) as the supercharge  $Q_1$ ; another self-adjoint supercharge is  $Q_2 = i\mathcal{R}Q_1$ . Therefore, for non-integer flux values, the translation symmetry of the Aharonov–Bohm system  $H_n^{AB}$  is broken, and corresponding mutually commuting generators  $\mathcal{P}_x$  and  $\mathcal{P}_y$  are substituted by nonlocal, mutually anticommuting, odd operators  $Q_1 = \mathcal{P}_x + i\epsilon\mathcal{R}\mathcal{P}_y$  and  $Q_2 = -\epsilon\mathcal{P}_y + i\mathcal{R}\mathcal{P}_x$ .<sup>14</sup>

By taking into account the dynamical conformal symmetry, the revealed hidden supersymmetric structure of the spinless Aharonov–Bohm system is extended to the superconformal  $osp(2|2)$  symmetry. By this superconformal symmetry, one can relate not only the states with the same value of the angular momentum and different values of the energy, see [45], but also the states with different energy values and different in one angular momentum in correspondence with figure 1 [53].

We have shown that the *hidden* superconformal symmetry of the spinless Aharonov–Bohm system is in one-to-one correspondence with the *explicit* center-of mass supersymmetric

<sup>14</sup> This picture can be compared loosely with that appearing in the BRST scheme of quantization of usual, non-supersymmetric gauge invariant theories, where after gauge fixing the even generators of gauge symmetries are substituted by the mutually anticommuting nilpotent BRST and anti-BRST operators [52].

structure of the system composed of the two two-anyon subsystems, the composites of one of which before switching on statistical interaction ( $\alpha = 0$ ) satisfy boson statistics, while another subsystem is formed by two identical fermion particles. The exotic models given by the Hamiltonians  $H_\alpha^\gamma$ ,  $\gamma = 0, \pi$ , with nontrivial behavior of the wavefunctions near the origin correspond in this interpretation to the case of anyons with a contact (zero-range) interaction.

The hidden supersymmetric structure is reflected in the scattering picture. To see this, consider the case of the Aharonov–Bohm model given by the Hamiltonian  $H_\alpha^{AB}$ . Its regular at the origin eigenfunctions, which correspond to a plane wave incident from the right ( $x = +\infty$ ,  $y = 0$ ), have a form [1, 8, 54]

$$\psi = \sum_{l=-\infty}^{\infty} a_l e^{il\varphi} \mathcal{J}_{|l+\alpha|}(kr), \tag{7.3}$$

$H_\alpha^{AB} \psi = k^2 \psi$ , where

$$a_l = e^{-i\frac{\pi}{2}|l+\alpha|}. \tag{7.4}$$

For the sake of definiteness, suppose that  $\alpha \in (-1, 0)$ . In this case, coefficients (7.4) satisfy the relation

$$a_{2l} = \varepsilon i a_{2l-1}, \quad \text{where } \varepsilon = \begin{cases} -1 & \text{for } l \geq 1, \\ +1 & \text{for } l \leq 0. \end{cases} \tag{7.5}$$

Acting on (7.3) with the supercharge (2.14), and taking into account the recurrence relations satisfied by the Bessel functions,

$$\mathcal{J}_{\nu \mp 1}(x) = \left( \pm \frac{d}{dx} + \frac{\nu}{x} \right) \mathcal{J}_\nu(x),$$

and relation (7.5), we find that the energy eigenfunctions (7.3) are simultaneously the supercharge eigenstates,  $Q_\alpha^{AB} \psi = -k \psi$ . The second supercharge (as well as the operator  $\mathcal{R}$ ) transforms the state (7.3) into another eigenstate of  $H_\alpha^{AB}$ , which corresponds to the plane wave incident from the left.

Making use of relation (7.5), energy eigenfunction (7.3) can be presented as a superposition of the supercharge eigenstates (2.19),

$$\psi = \sum_{l=-\infty}^0 \Phi_l^- + \sum_{l=1}^{+\infty} \Phi_l^+, \tag{7.6}$$

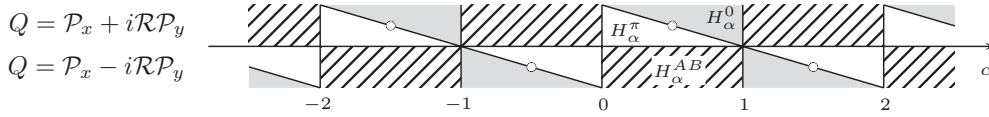
where

$$\Phi_l^-(r, \varphi) = e^{i\frac{\pi}{2}\alpha} (-1)^l e^{2il\varphi} (\mathcal{J}_{-(2l+\alpha)}(kr) - i e^{-i\varphi} \mathcal{J}_{-(2l-1+\alpha)}(kr)), \tag{7.7}$$

$$\Phi_l^+(r, \varphi) = e^{-i\frac{\pi}{2}\alpha} (-1)^l e^{2il\varphi} (\mathcal{J}_{2l+\alpha}(kr) + i e^{-i\varphi} \mathcal{J}_{2l-1+\alpha}(kr)), \tag{7.8}$$

$Q_\alpha^{AB} \Phi_l^- = -k \Phi_l^-$ ,  $l = 0, -1, -2, \dots$ ,  $Q_\alpha^{AB} \Phi_l^+ = -k \Phi_l^+$ ,  $l = 1, 2, \dots$ . The energy eigenstate  $\mathcal{R}\psi(r, \varphi) = \psi(r, \varphi + \pi)$ , that corresponds to the plane wave incident from the left, is the eigenstate of the supercharge of the eigenvalue  $+k$ ,  $Q_\alpha^{AB} \psi(r, \varphi + \pi) = +k \psi(r, \varphi + \pi)$ . The superpositions  $\psi(r, \varphi) \pm i \psi(r, \varphi + \pi)$  are the eigenstates of the second supercharge  $Q_2 = i\mathcal{R}Q_\alpha^{AB}$ ,  $Q_2(\psi(r, \varphi) \pm i \psi(r, \varphi + \pi)) = \mp k(\psi(r, \varphi) \pm i \psi(r, \varphi + \pi))$ . For  $\alpha = -1/2$ , the states (7.6) and  $\mathcal{R}\psi$  form the invariant subspace also for two additional nonlocal integrals of motion that appear in the system in this case,  $\tilde{\mathcal{R}}_x = -i e^{i\varphi} \mathcal{R}_x$ ,  $\tilde{\mathcal{R}}_y = e^{i\varphi} \mathcal{R}_y$ , where  $\mathcal{R}_x : \varphi \rightarrow \pi - \varphi$ ,  $\mathcal{R}_y : \varphi \rightarrow -\varphi$ .

The nonphysical nature of the operators  $\mathcal{P}_x$  and  $\mathcal{P}_y$  can be revealed immediately if we apply them to the Hamiltonian eigenfunction (7.3). The action of the operator  $\mathcal{P}_x + i\mathcal{P}_y$



**Figure 2.** The figure illustrates the three different self-adjoint extensions of  $H_\alpha$  in dependence on  $\alpha$ . Upper and lower cases correspond to different definitions (2.5) and (3.1) of the supercharges. Rectangular shaded zones correspond to the setting discussed by Aharonov and Bohm, while gray and white triangular zones correspond to the exotic models represented by  $H_\alpha^0$  and  $H_\alpha^\pi$  respectively. The circles for half-integer values of  $\alpha$  indicate that  $H_\alpha^0$  and  $H_\alpha^\pi$  are unitarily equivalent in this case, see section 4.

produces a state, in which the  $l = 1$  partial wave is multiplied by the function  $\mathcal{J}_{|\alpha|-1}(kr)$  that has a not permitted, singular behavior at the origin. Analogously, the state  $(\mathcal{P}_x - i\mathcal{P}_y)\psi$  contains a partial wave with  $l = 0$  multiplied by the singular at the origin function  $\mathcal{J}_{-|\alpha|}(kr)$ . The supercharge (2.14) can be written in the form  $Q_\alpha^{AB} = \Pi_+(\mathcal{P}_x + i\mathcal{P}_y) + \Pi_-(\mathcal{P}_x - i\mathcal{P}_y)$ . Its projectors on the subspaces with even and odd  $l$ ,  $\Pi_\pm = \frac{1}{2}(1 \pm \mathcal{R})$ , just annul the singularities produced by nonphysical operators  $\mathcal{P}_x \pm i\mathcal{P}_y$  in corresponding partial waves. One can show that in the case of the exotic systems  $H_\alpha^\gamma$ ,  $\gamma = 0, \pi$ , considered in section 3, the picture is similar: the operators  $\mathcal{P}_x \pm i\mathcal{P}_y$  acting on the states of the domain of the Hamiltonian  $H_\alpha^\gamma$ , in contrast with the action of the supercharges, produce the states that do not belong to the domain. This explains the mechanism of translation symmetry breaking, and its substitution for the hidden supersymmetry, as well as a purely formal character of factorization (7.1). Note also here that in the case  $\alpha = n$ , the action of the operators  $\mathcal{P}_x \pm i\mathcal{P}_y$  on the energy eigenstates (7.3) does not produce singularities, and operators  $\mathcal{P}_x$  and  $\mathcal{P}_y$  commute on the domain of the Hamiltonian  $H_n^{AB}$ . This corresponds to a unitary equivalence of the model  $H_n^{AB}$  to a free planar particle system discussed in section 2.

Partial wave analysis applied to the wavefunction (7.3) gives the scattered wave with asymptotic behavior for large  $r$ , see [1, 54],  $\psi_{sc} \rightarrow r^{-1/2} e^{ikr} f(\varphi)$ ,

$$f(\varphi) = (2\pi ik)^{-1/2} \sum_{l=-\infty}^{+\infty} e^{il(\varphi-\pi)} (e^{2i\delta_l} - 1),$$

where the phase shifts are given by

$$\delta_l = -\frac{\pi}{2}|l + \alpha| + \frac{\pi}{2}|l|. \tag{7.9}$$

By taking into account (7.4) and (7.5), we get the relation

$$e^{2i\delta_{2l}} = e^{2i\delta_{2l-1}}. \tag{7.10}$$

This relation between the phase shifts reflects coherently with the picture presented in figure 1, a hidden supersymmetry in the scattering problem of the spinless Aharonov–Bohm model in the case  $\alpha \in (-1, 0) \bmod 2$ . In the case  $\alpha \in (0, 1) \bmod 2$ , index  $2l - 1$  on the right-hand side of relation (7.10) is changed for  $2l + 1$  in correspondence with figure 2.

Finally, we note that the original Aharonov–Bohm calculation of the scattering amplitude [1], mathematically more justified in comparison with partial wave analysis, see [54], was based on the separation of the wavefunction (7.3) into three functions:  $\psi = \psi_1 + \psi_2 + \psi_3$ . In the case  $\alpha \in (-1, 0)$ , this corresponds to the separation of the wavefunction  $\psi$  in a partial wave with  $l = 0$  ( $\psi_3$ ), and in the infinite sums with  $l > 0$  ( $\psi_1$ ) and  $l < 0$  ( $\psi_2$ ) [1, 54]. For the function  $\psi_1$  the equivalent integral representation was found in [1], that allowed the authors to find its asymptotic expansion, and then to calculate the scattering amplitude. The function

$\psi_1$  is nothing else in the second series in (7.6). This means that the original method used in [1] is coherent with the hidden supersymmetric structure revealed in the present paper.

In conclusion, let us discuss some open problems that would be interesting for further investigation.

The stationary Schrödinger equation of the Aharonov–Bohm model is separable in polar coordinates. Its radial equation corresponds to stationary Schrödinger equation of Calogero model. When we specify the self-adjoint extension of the formal Hamiltonian operator  $H_\alpha$ , the self-adjoint extension of the radial part of  $H_\alpha$  is fixed as well. In other words, fixing the value of the angular momentum, the (rotationally invariant) self-adjoint extension of  $H_\alpha$  fixes the self-adjoint extensions of the Calogero model [55]. In [56], Gitman *et al* discussed recently the dilatation symmetry of the self-adjoint extensions of this one-dimensional system. They concluded that there are only few self-adjoint extensions of the Calogero model which possess scale invariance. We described three Aharonov–Bohm type systems, represented by  $H_\alpha^{AB}$  and  $H_\alpha^\gamma$ ,  $\gamma = 0, \pi$ . These systems proved to be scale invariant. It is a quite intriguing question, whether these two distinct symmetries, scale invariance and hidden supersymmetry, are interrelated somehow. We suppose that this is indeed the case. Verification of this hypothesis could provide a deeper insight into the physical system and its symmetries as well.

Recently, the hidden supersymmetry of the reflectionless Pöschl–Teller system was explained in [57] in the context of non-relativistic AdS/CFT correspondence [58, 59]. The rather natural question is then whether some AdS/CFT holography interpretation exists for the hidden superconformal symmetry observed here.

The Aharonov–Bohm type systems described formally by  $H_\alpha$ , can have up to two bound states. The systems with negative energies were disqualified in our framework from the very beginning by requirement of the presence of a self-adjoint supercharge. This is in correlation with spontaneous breakdown of their scale invariance. However, such systems could fit into the framework of the nonlinear supersymmetry. Analysis of this possibility requires a separate consideration.

We analyzed the spinless particle case. It would be interesting to consider the systems with spin degrees of freedom as well [53]. The spin one-half system would be governed by the Pauli Hamiltonian, whose diagonal components would differ formally just in the sign of the magnetic field, cf [50, 60]. This suggests that the actual self-adjoint extensions of the upper and the lower diagonal elements of the matrix Hamiltonian could differ in some way. The standard supersymmetry should be present then in addition to the hidden supersymmetry, at least in some particular cases. The presence of both, explicit and hidden, supersymmetries should give rise to the structure of tri-supersymmetry [26, 53, 61].

As we observed in section 4, in the case of half-integer flux values there exists a three-parametric family of unitary transformations (4.5), generated by  $\mathcal{R}$ ,  $\tilde{\mathcal{R}}_x$  and  $\tilde{\mathcal{R}}_y$ . These transformations do not change the formal Hamiltonian  $H_\alpha$ , but interchange its self-adjoint extensions. Hence, there exists a three-parametric family of self-adjoint extensions of  $H_\alpha$  which allow the existence of the hidden supersymmetry, see (4.14). We discussed a few particular cases in (4.12), where the systems associated with  $\tilde{Q}_{1/2}^\gamma$  for  $\gamma \in \{0, \pi/2, \pi, 3\pi/2\}$  were interrelated by these unitary mappings. The family of all the self-adjoint extensions of  $H_\alpha$  is four parametric [31]. So it seems that a great part of the self-adjoint extensions of  $H_\alpha$  possess hidden supersymmetry for semi-integer values of  $\alpha$ . It would be interesting to clarify this point.

We investigated the question of the presence of the hidden supersymmetry in spinless quantum mechanical Aharonov–Bohm type systems. The intriguing open question is whether such a symmetry may be present in related field systems. The simplest system for such a generalization could be a non-relativistic  $(2+1)$ -dimensional model of a boson field minimally

coupled to a Chern–Simons field [35, 62, 63]. If the hidden bosonized supersymmetry of the nature discussed here is present in such a field system, then its supersymmetrically extended (by inclusion of a fermion field) version [48] would be described more readily than the  $osp(2|2)$  superconformal structure [49], related to the tri-supersymmetry [26, 61].

### Acknowledgments

The work has been partially supported by DICYT (USACH), MECESUP Project FSM0605, CONICYT and FONDECYT grants 1050001, 7080056, 1095027, 3085013 and 3100123, Chile, by the MSMT ‘Doppler Institute’ project LC06002, and by CONICET (PIP 01787) and UNLP (Proy. 11/X492), Argentina. HF is indebted to the Physics Department of Santiago University (Chile) for hospitality. The Centro de Estudios Científicos (CECS) is funded by the Chilean Government through the Millennium Science Initiative and the Centers of Excellence Base Financing Program of Conicyt. CECS is also supported by a group of private companies which at present includes Antofagasta Minerals, Arauco, Empresas CMPC, Indura, Naviera Ultragas and Telefónica del Sur.

### Appendix A

Let us present in more detail the procedure of self-adjoint extension of the following operator:

$$\hat{Q} = \mathcal{P}_x + i\mathcal{R}\mathcal{P}_y.$$

The supercharge (2.5) coincides with this operator for  $\alpha \in [-1, 0] \bmod 2$ .  $\hat{Q}$  can be identified with (3.1) for  $\alpha \in (0, 1) \bmod 2$  as well. Hence, the analysis of self-adjoint extensions of  $\hat{Q}$  for any value of the flux will provide, using the unitary transformation  $U_1$  consequently, a complete information on self-adjoint extensions of both (2.5) and (3.1).

The symmetric operator  $Q$  is a restriction of  $\hat{Q}$  to  $C_0^\infty(\mathbb{R}^2 - \{0\})$ . The following relation will be useful in the forthcoming analysis:

$$(\phi, \hat{Q}\psi) - (\hat{Q}\phi, \psi) = \lim_{r \rightarrow 0^+} \int_0^{2\pi} d\varphi r [(-i \cos \varphi + R \sin \varphi)\phi(r, \varphi)]^* \psi(r, \varphi). \quad (\text{A.1})$$

One can easily see that  $Q$  is symmetric, since the right-hand side of equation (A.1) vanishes for all  $\phi, \psi \in \mathcal{D}(Q)$ .

*The adjoint.* The adjoint of  $Q$ ,  $Q^\dagger$ , is a linear operator defined on the set of those functions for which  $(\phi, \hat{Q}\psi)$  is a linear continuous functional of  $\psi \in \mathcal{D}(Q)$  (see [30], for example). This requires that for any  $\phi \in \mathcal{D}(Q^\dagger)$  there is a vector  $\chi \in \mathbf{L}_2(\mathbb{R}^2)$  such that

$$(\phi, Q\psi) = (\chi, \psi), \quad \forall \psi \in \mathcal{D}(Q). \quad (\text{A.2})$$

For each  $\phi$ , this vector is unique (since  $\mathcal{D}(Q)$  is dense in  $\mathbf{L}_2(\mathbb{R}^2)$ ) and the action of the adjoint operator is defined as  $Q^\dagger\phi := \chi$ .

Since functions  $\psi(r, \varphi) \in \mathcal{D}(Q)$  identically vanish in some neighborhood of the origin, the right-hand side of equation (A.1) vanishes for any function  $\phi(r, \varphi)$  such that  $\hat{Q}\phi(r, \varphi) \in \mathbf{L}_2(\mathbb{R}^2)$ . Therefore, the adjoint operator is densely defined in

$$\mathcal{D}(Q^\dagger) = \{\phi(r, \varphi) \in \mathcal{AC}(\mathbb{R}^2 \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^2) : \hat{Q}\phi(r, \varphi) \in \mathbf{L}_2(\mathbb{R}^2)\}, \quad (\text{A.3})$$

where  $\mathcal{AC}(\mathbb{R}^2 \setminus \{0\})$  is a set of absolutely continuous functions in punctured plane [30].

Since the set  $\{e^{im\varphi}, m \in \mathbb{Z}\}$  is a complete orthogonal system in  $\mathbf{L}_2(S^1)$ , we can write

$$\phi(r, \varphi) = \sum_{m \in \mathbb{Z}} e^{im\varphi} \phi_m(r), \quad (\text{A.4})$$

where  $\phi_m(r) \in \mathcal{AC}(\mathbb{R}^+ \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^+; r dr)$ . Then the condition  $\hat{Q}\phi(r, \varphi) \in \mathbf{L}_2(\mathbb{R}^2)$  for  $\alpha \notin \mathbb{Z}$  reduces to

$$|\phi_{2l}(r)| = \begin{cases} O(1), & \text{for } 2l + \alpha \notin (0, 1), \\ O(r^{-(2l+\alpha)}), & \text{for } 2l + \alpha \in (0, 1), \end{cases} \quad (\text{A.5})$$

and

$$|\phi_{2l-1}(r)| = \begin{cases} O(1), & \text{for } 2l - 1 + \alpha \notin (-1, 0), \\ O(r^{(2l-1+\alpha)}), & \text{for } 2l - 1 + \alpha \in (-1, 0). \end{cases} \quad (\text{A.6})$$

For  $\alpha = \beta - 2l_0 \in \mathbb{Z}$ ,  $\beta \in \{0, 1\}$ , the partial waves  $\phi_j$  are subject to the following restrictions:

$$|\phi_j| = O(1) \quad \text{for } j \neq 2l_0 - \beta, \quad |\phi_{2l_0-\beta}| = O(\sqrt{-\log \mu r}). \quad (\text{A.7})$$

The closure  $\overline{Q}$ . The minimal closed extension of  $Q$  is called the *closure* of this operator, which is defined as  $\overline{Q} := (Q^\dagger)^\dagger$ . According to the previous discussion on the definition of the adjoint operator and equation (A.1), it follows that its domain is the set of functions  $f(r, \varphi)$  for which  $\hat{Q}f(r, \varphi) \in \mathbf{L}_2(\mathbb{R}^2)$  and (see (A.1))

$$\lim_{r \rightarrow 0^+} \int_0^{2\pi} d\varphi r [(-i \cos \varphi + R \sin \varphi) f(r, \varphi)] \phi(r, \varphi)^* = 0, \quad \forall \phi(r, \varphi) \in \mathcal{D}(Q^\dagger). \quad (\text{A.8})$$

To get an insight into the restrictions on  $f(r, \varphi)$  posed by this requirement, it is convenient to employ the Fourier series of  $f(r, \varphi)$ ,

$$f(r, \varphi) = \sum_{m \in \mathbb{Z}} e^{im\varphi} f_m(r), \quad f_m(r) \in \mathcal{AC}(\mathbb{R}^+ \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^+; r dr). \quad (\text{A.9})$$

For  $\alpha \notin (0, 1) \bmod 2$ , the conditions posed on  $f_m$  are identical to (A.5) and (A.6) (resp. (A.7)). This means that the domains of definition  $Q^\dagger$  and  $\overline{Q}$  are identical and the operator  $Q$  is essentially self-adjoint<sup>15</sup> in this case. Having in mind the note in the beginning of the appendix, we conclude that the operator  $Q_\alpha$  defined in (2.5) has a unique self-adjoint extension  $Q_\alpha^{AB}$  for any value of the flux. Its domain of definition can be written as

$$\mathcal{D}(Q_\alpha^{AB}) = \left\{ f(r, \varphi) = \sum_l f_l(r) e^{il\varphi}, f_l \in \mathcal{AC}(\mathbb{R}^+ \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^+ l; r dr), |f_l(r)| = O(1) \right\} \quad \text{for } \alpha \notin \mathbb{Z}, \quad (\text{A.10})$$

and for  $\alpha = -2l_0 + \beta \in \mathbb{Z}$ ,

$$\mathcal{D}(Q_{-2l_0+\beta}^{AB}) = \left\{ f(r, \varphi) = \sum_l f_l(r) e^{il\varphi}, f_l \in \mathcal{AC}(\mathbb{R}^+ \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^+ l; r dr), \right. \\ \left. |f_l(r)| = O(1) \text{ for } m \neq -2l_0 + \beta, |f_{2l_0-\beta}| = O(\sqrt{-\log r}) \right\}. \quad (\text{A.11})$$

For  $2l_0 + \alpha \in (0, 1)$ , the conditions on  $f_{2l_0}$  and  $f_{2l_0-1}$  are more restrictive,

$$f_{2l_0}(r) = o(r^{-(2l_0+\alpha)}), \quad f_{2l_0-1}(r) = o(r^{(2l_0-1+\alpha)}). \quad (\text{A.12})$$

This means that the restriction of  $Q^\dagger$  to the subspace  $\mathcal{H}_{l_0}$  has a larger domain than the restriction of  $\overline{Q}$  to this subspace. Since these domains do not coincide,  $Q$  is not essentially self-adjoint. Let us remind that for these values of  $\alpha$ ,  $Q$  corresponds to  $\tilde{Q}_\alpha$ , see (3.1).

*Deficiency subspaces.* We will find solutions  $\phi = \phi_{2l} e^{2li\varphi} + \phi_{2l-1} e^{(2l-1)i\varphi}$  of  $Q^\dagger \phi = \pm i\mu \phi$  for  $\phi \in \mathcal{D}(Q^\dagger)$ . We can use directly equation (2.18) for  $\lambda = \pm i\mu$ . It reduces to

$$\phi_{2l}''(r) + \frac{1}{r} \phi_{2l}'(r) - \left\{ \mu^2 + \frac{(2l + \alpha)^2}{r^2} \right\} \phi_{2l}(r) = 0. \quad (\text{A.13})$$

<sup>15</sup> A densely defined symmetric operator  $A$  is essentially self-adjoint if  $\overline{A} = A^\dagger$ .



This differential equation has solutions of the form  $\phi_{2l}(r) = C_1 K_{|2l+\alpha|}(\mu r) + C_2 I_{|2l+\alpha|}(\mu r)$ , where  $I_\nu$  and  $K_\nu$  are the modified Bessel functions of the first and second (or Macdonald function) kinds, respectively. The modified Bessel function of the first kind ( $I_\nu$ ) has to be discarded as it diverges for  $f \rightarrow +\infty$ ,  $C_2 = 0$  for all  $l$ . The function  $K_\nu$  decreases exponentially in infinity. For  $r \rightarrow +0$ , it reads

$$K_\nu(z) \sim 2^{|\nu|-1} \Gamma(|\nu|) z^{-|\nu|} (1 + O(z^2)). \tag{A.14}$$

We require the eigenvectors of  $Q^\dagger$  to lie in  $\mathcal{D}(Q^\dagger)$  and to be square integrable in particular. This requirement is met only for  $0 < 2l + \alpha < 1$ , i.e. for  $l = l_0$ . Then there is one (and only one) eigenvector of  $Q^\dagger$  corresponding to each of the eigenvalues  $\lambda = \pm i\mu$ , given by

$$\Phi_\pm = e^{i2l_0\varphi} K_{2l_0+\alpha}(\mu r) \pm e^{i(2l_0-1)\varphi} K_{1-(2l_0+\alpha)}(\mu r). \tag{A.15}$$

In the main text, we fixed the scale parameter  $\mu = 1$  without lost of generality. Note that  $\|\Phi_+\| = \|\Phi_-\|$ .

*Self-adjoint extensions.* Hence, the *deficiency subspaces*  $\mathcal{K}_\pm$  are one-dimensional for  $\alpha \in (0, 1) \bmod 2$ . We remind that  $Q$  coincides formally with  $\tilde{Q}_\alpha$  (defined in (3.1)) for this value of the magnetic flux. The *deficiency indices* are equal to one,  $n_\pm := \dim \mathcal{K}_\pm = 1$ , and, according to von Neumann’s theory of self-adjoint extensions of symmetric operators [30], the self-adjoint extensions of  $\tilde{Q}_\alpha$  are characterized by the isometries  $\mathcal{K}_+ \rightarrow \mathcal{K}_-$  (which, in the present case, form a group  $U(1)$  whose elements correspond to a phase factor  $e^{i\gamma}$ ). Let us denote these self-adjoint extensions by  $Q_\alpha^\gamma$ . Their domain of definition has the following form:

$$\begin{aligned} \mathcal{D}(Q_\alpha^\gamma) &:= \{ \Phi(r, \varphi) = f(r, \varphi) + A[\Phi_+(r, \varphi) + e^{i\gamma} \Phi_-(r, \varphi)] : \\ & f(r, \varphi) \in \mathcal{D}(\overline{Q}), A \in \mathbb{C}, \gamma \in [0, 2\pi) \}. \end{aligned} \tag{A.16}$$

The domain of definition of  $\overline{Q}$  for these values of the flux is given by (A.4), (A.5), (A.6) and (A.12). The operator  $Q_\alpha^\gamma$  acts as

$$Q_\alpha^\gamma \Phi(r, \varphi) := Q^\dagger \Phi(r, \varphi) = \overline{Q} f(r, \varphi) + i\mu A[\Phi_+(r, \varphi) - e^{i\gamma} \Phi_-(r, \varphi)]. \tag{A.17}$$

Taking into account (A.15), the domain can be written as

$$\mathcal{D}(Q_\alpha^\gamma) = \{ f(r, \varphi) + A(K_{2l_0+\alpha} e^{2l_0i\varphi} (1 + e^{i\gamma}) + K_{1-2l_0-\alpha} e^{(2l_0-1)i\varphi} (1 - e^{i\gamma})) \}, \tag{A.18}$$

where  $f(r, \varphi)$  is from  $\mathcal{D}(\overline{Q})$ .

## Appendix B

Let us take  $\gamma$  as a free parameter. The Hamiltonian  $H_\alpha^\gamma$  is self-adjoint as it is a square of self-adjoint supercharge  $\tilde{Q}_\alpha^\gamma$ . We can define the domain  $\mathcal{D}_c$ , see (5.4), for the current extension  $H_\alpha^\gamma$ . It is dense in  $L_2(\mathbb{R}^2)$  as it contains infinitely smooth functions with compact support as well.

However,  $\mathcal{D}_c$  cannot accommodate the wave packets (normalizable combinations of scattering states) for general value of  $\gamma$ . Let us demonstrate this in the following way: we restrict  $\alpha \in [0, 1)$ . Let  $\Phi_0(r, \varphi) = \phi_0(r) + \phi_{-1} e^{-i\varphi}$  be a function lying in the intersection of  $\mathcal{H}_0$ ,  $\mathcal{D}(\tilde{Q}_\alpha^\gamma)$  and  $\overline{\mathcal{D}}$ . It has the asymptotic behavior at the origin prescribed by (3.9). Acting with  $\overline{\mathcal{D}}$  we get

$$\overline{\mathcal{D}}\phi_0(r) \sim A(1 - \alpha)(1 + e^{i\gamma}) \frac{\Gamma(\alpha)}{2^{1-\alpha}} r^{-\alpha} (1 + O(r^2)), \tag{B.1}$$

$$\overline{D}\phi_{-1}(r) \sim A\alpha(1 - e^{i\gamma}) \frac{\Gamma(1 - \alpha)}{2^\alpha} r^{-1+\alpha} (1 + O(r^2)). \quad (\text{B.2})$$

We require that the resulting function does not leave the domain of definition of  $Q_\alpha^\gamma$ . It is a necessary condition to keep the wave packets composed of scattering states from  $\mathcal{H}_0$  within  $\mathcal{D}_c$ . Considering  $\alpha$  as a free parameter, this requirement can be satisfied just for  $\gamma = 0$  or  $\gamma = \pi$ . There exists another possibility as well: when  $\alpha = 1/2$ ,  $\overline{D}\Phi_0$  satisfies (3.9) for any value of  $\gamma$ . This is in agreement with our observation of section 4, where the broader family of the self-adjoint extensions with hidden supersymmetry generator  $\mathcal{U}\tilde{Q}_{1/2}^0\mathcal{U}^{-1}$  was revealed in the case of the half-integer flux. In particular, we discussed the systems associated with  $\tilde{Q}_{1/2}^{\pi/2}$  and  $\tilde{Q}_{1/2}^{3\pi/2}$ .

Let us present here the domains of definitions of the operators  $D$ ,  $S_1$ ,  $F$  and  $\mathcal{Z}$  :

$$\mathcal{D}(\overline{D}) = \mathcal{D}(D^\dagger) = \{\psi(r, \varphi) \in \mathcal{AC}(\mathbb{R}^2 \setminus \{0\}) \cap L_2(\mathbb{R}^2) | r\partial_r\psi(r, \varphi) \in L_2(\mathbb{R}^2)\}, \quad (\text{B.3})$$

$$\mathcal{D}(S_2) = \mathcal{D}(S_1) = \{\Psi \in L_2(\mathbb{R}^2) : S_1\Psi \in L_2(\mathbb{R}^2)\}, \quad (\text{B.4})$$

$$\mathcal{D}(F) = \{\Psi \in L_2(\mathbb{R}^2) : F\Psi \in L_2(\mathbb{R}^2)\}, \quad (\text{B.5})$$

$$\mathcal{D}(\mathcal{Z}) = \{\Psi \in L_2(\mathbb{R}^2) : \mathcal{Z}\Psi \in L_2(\mathbb{R}^2)\}. \quad (\text{B.6})$$

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